The Method of Determining the Priority of Candidates by Means of Preferential Voting Based on an Algebraic Approach

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Introduction

In various subject areas, practical situations do not require a formal straight-line solution, but can be successfully solved by applying special procedures. Such procedures smooth out the initial problem, pre-harmonize the views of various persons who have an influence on decision-making.

Participants in the decision-making process pre-evaluate the possible solutions to the problem and thus the entire team of experts prepares for a compromise solution in advance. Such procedures can not be spontaneous or introduced by a spontaneous method: they should be prepared in advance, formalized, substantiated and researched.

Preferential voting and the relevance of the issue

Preferential voting is a system of elections in which a voter has the option to rank candidates in order of preference. This research direction is one of the key aspects in proportional electoral systems with open-list PR systems and flexible-list PR systems.

The distinctive feature of the preferential voting system is that a voter does not vote for just one candidate but for several candidates, indicating their preferences for whom they would like to see elected in the first place, second place, and so on. The institution of preferential voting aims to provide voters with the opportunity not only to vote for a party's list of candidates but also to express preferences within that list in favor of specific candidates, thereby promoting their election.

Preferential voting around the globe

Preferential voting is used in various countries around the world, including: Australia, Austria, Belgium, Denmark, Estonia, India, Ireland, Italy, Malta, Netherlands, New Zealand, Norway, Poland, Slovakia, Czech Republic, Sweden, Sri Lanka and other countries.

Preferential voting allows for a more precise expression of the voters will and promotes the representation of a broader spectrum of political opinions in parliament or other governing bodies. Therefore, research on issues related to preferential voting is highly relevant.

Problem Statement

Let k experts conduct a preliminary examination of n alternatives, alternatives or candidates for the position from the set A. According to the terms of the examination, members of the collective at the first stage should set multiple comparisons of v alternatives, where v<<n. An additional requirement may be a fixed value v for all experts. The upper limit of v is determined by the psycho-physiological capabilities of a person. It can not be greater, for example, than 9, given the Miller number equal 7+\-2.

We will denote subset of applicants that chosen by i-th member as $A_i^v, i \in I = \{1,...,k\}$,
On the initial stage number of expert and alternatives is equal k=n.
Without reducing the generality, we will assume that $A_j^v = \{a_{i_1}^j \succ ... \succ a_{i_2}^j \succ ... \succ a_{i_v}^j\}$, (1)
for all members $j \in I$.

Union of Subsets

In the first stage of solving the problem, the union of subsets of multiple comparisons of alternatives provided by experts is carried out into a single set A that includes all the measures proposed by the experts

$$a_i \in A, i = 1,..., n,$$

$$A = \bigcup_{i=1}^{k} A_i. \qquad (2)$$

Clearly, various variants of relationships between subsets are allowed in this process.

$$A_{i_1} \cup A_{i_2} = \varnothing, \quad A_{i_1} \cup A_{i_2} \neq \varnothing, \quad A_{i_1} = A_{i_2}, \quad i_1, i_2 \in J.$$

It is also necessary to introduce heuristics for aggregating (smoothing) individual multiple comparisons. After applying these heuristics, the set (2) of combined subsets is narrowed down in order to reduce the dimensionality of the problem.

Heuristic H1: There is a single collective, i.e., a graph that corresponds to the preferences of the members of collective, as identified based on previous voting, although this does not exclude a situation of multipolar leadership.

Algebraic Approach

Methods for processing expert information can be divided into three main groups: Statistical methods, Scaling methods, Algebraic methods

The essence of algebraic methods lies in defining a distance on the set of admissible evaluations, and the resulting evaluation is determined as the one for which the distance to the evaluations of experts, according to a certain chosen criterion, is minimal. Based on the algebraic approach, we will determine the coefficients of relative competence of experts.

One important way to represent expert information is the ranking of alternatives. In many practical problems, there is a need to use multiple comparisons or incomplete rankings. In such cases, classical methods from voting theory have been applied to determine the generalized ranking of alternatives. However, in these cases, it is promising to use algebraic methods for computing the median, which reflects the collective opinion of experts.

The expert primarily reveals their preferences, structure, and relationships between alternatives in their perception, as well as the prospects of leadership within the collective.

The set of all given experts in the form of alternative (1) is the domain of admissible solutions for determining the result of ranking the alternative R*.

Distances between alternative positioning are produced using the metric:
– Cook's metric of dissimilarity of ranks (place, position) alternative,
– Cook's metric of dissimilarity of ranks of alternatives in individual locations

$$d\left(R^j, R^l\right) = \sum_{i \in I} \left| r_i^j - r_i^l \right|, \qquad (4)$$

where r_i^l is the rank of the i-th alternative in the ranking of the l-th expert $R^l, l \in L,\ 1 \le r_i^l \le n,$
– Hamming metrics

$$d\left(B^j, B^l\right) = 0{,}5 \sum_{i \in I} \sum_{s \in I} \left| b_{is}^j - b_{is}^l \right|, \qquad (5)$$

where $B^l = \left(b_{is}^l\right),\ l \in L,\ i,s \in I,$– is the matrix of pairwise comparisons (MPC) that provides the ranking $R^l, l \in L;$
– Euclid metrics
– the preference vector, the elements of which are the number of alternatives that precede each alternative in the ranking

Criteria that are most often used in such cases:
– additive
– minimax

The most common method of finding the resulting ranking of alternatives is the calculation of the median of the given rankings. We denote the set of all possible rankings n of alternatives by Ω^R , and the set of MPCs corresponding to all possible rankings n of alternatives by Ω^B . The set of rankings given by experts will be denoted by R^A , and the set of MPCs corresponding to them by R^B .

For the case considered in this work, the power of the sets R^A and R^B is the same $\left|R^A\right| = \left|R^B\right| = n, \quad R^l \in R^A, \ B^l \in R^B_{\cdot}, l \in L$ It's understandable that $R^A \subset \Omega^R, \ R^B \subset \Omega^B$In the general case, the power of the set $\left|\Omega^B\right| = 2^{n(n-1)/2}$. But for the method described in this paper, we will assume that $\left|\Omega^R\right| = \left|\Omega^B\right| = n!$ since we are not interested in non-transitive elements of the solution space Ω^B

For the Cook metric (the metric of the discrepancy in the ranks of alternatives) of type (2) when using the utilitarian criterion, the Cook-Sayford median is calculated:

$$R^{CS} \in \Omega^{CS} = Arg \min_{R \in \Omega^R} \sum_{l \in L} d\left(R, R^l\right). \quad (6)$$

When using the egalitarian criterion, the GV-median (compromise) is calculated:

$$R^{ГВ} \in \Omega^{ГВ} = Arg \min_{R \in \Omega^R} \max_{l \in L} d\left(R, R^l\right). \quad (7)$$

For the Hamming metric (3) when using the utilitarian criterion, the Kemeny-Snell median is calculated:

$$R^{KC} \in \Omega^{KC} = Arg \min_{B \in \Omega^B} \sum_{l \in L} d\left(B, B^l\right). \quad (8)$$

When using the egalitarian criterion, the VG-median (compromise) is calculated:

$$R^{BГ} \in \Omega^{BГ} = Arg \min_{B \in \Omega^B} \max_{l \in L} d\left(B, B^l\right). \quad (9)$$

In addition, the competence coefficients of experts can be taken into account in such tasks: $\rho_1, ..., \rho_k$

To determine the agreed ranking of candidates, we will introduce additional heuristics.

Heuristics H3 (maximum satisfaction of desires). It means that the expert wants all the alternatives in his multiple comparison to become winners in the sequence he specified, that is, he shows a desire to win as much as possible.

Thus, the desire of each expert appears to be utopian. But given the fact that they are all in the same conditions, the situation is not so idealized.

Let the conditions of the previous preferential voting determine that the expert set the multiple comparisons in the form $a_{i_1} \succ a_{i_2} \succ a_{i_3}$ where $i_1, i_2, i_3 \in I.$

Then it is easy to see that when in the options from which a certain ranking is chosen, depending on the sequence of indices of the alternatives of the same three, it is possible to determine the distances for choosing a compromise ranking.

For sequences $a_{i_1} \succ \ldots \succ a_{i_3} \succ \ldots \succ a_{i_2}$ and $a_{i_2} \succ \ldots \succ a_{i_1} \succ \ldots \succ a_{i_3}$ distances are equal 2.

For sequences $a_{i_2} \succ \ldots \succ a_{i_3} \succ \ldots \succ a_{i_1}$, $a_{i_3} \succ \ldots \succ a_{i_1} \succ \ldots \succ a_{i_2}$, and $a_{i_3} \succ \ldots \succ a_{i_2} \succ \ldots \succ a_{i_1}$ distances are equal 4. $d_M^l = \left|1 - r_{i_1}^l\right| + \left|2 - r_{i_2}^l\right| + \left|3 - r_{i_3}^l\right|, \; l \in I.$

Heuristic H4 (moderate reciprocity). That is, each member of collective in conditions of fair voting understands that not everyone can be a winner, that the probability of satisfying all wishes is negligible. Moreover, not one can be defeated, but at least the individual order set by each member should be preserved: it is important for victory.

In this case, the distance from the given individual multiple comparisons to the resulting ranking $R^0 = a^0_{i_1} \succ a^0_{i_2} \succ ... \succ a^0_{i_v}$ is determined as follows: $d^l_V = \left|r^0_{i_1} - r^l_{i_1}\right| + \left|r^0_{i_2} - r^l_{i_2}\right| + \left|r^0_{i_3} - r^l_{i_3}\right|. \ l \in I.$

Note that at v=4, we have 3 member located at a distance of 2 from the possible configuration of candidates in the resulting ranking, 7 members at distance 4, 9 members at distance 6 and 4 members at distance 8.

At v=5, the distance distribution has the following indicators, respectively. 4 members - 2, 12 members - 4, 24 members - 6, 35 members - 8, 24 members - 10, 20 members - 12.

A flexible approach to the ranking of alternatives, when each expert can propose a partial ranking of a subset chosen by him from the entire set of alternatives, and the search for a complete resulting ranking of alternatives by algebraic methods were not proposed by the researchers. After all, it becomes possible to take into account the situation when some experts may not know about the peculiarities of the functioning of individual elements from a set of alternatives.

- Transport logistics: the sequence of visiting nodes on the route;
- Reverse warehouse logistics: the sequence of unloading goods from a warehouse or transport;
- Trust (trust society): prioritization of investing funds in packages of shares;
- Purchase of cars or other equipment (thermal imagers, Kevlar helmets) for the Armed Forces of Ukraine;
- Preliminary elections of the head of the department;
- Problem about the backpack: the importance of having items;
- Application for honors by quota;
- Formation of a track of Ukrainian songs;
- Formation of a library with limitations in space;
- Nomination of candidates to the Student Parliament;

- Creation of a restaurant and preliminary study of demand with the help of a focus group: planning a restaurant menu is an important element of the hotel and restaurant business;
- Forming a repertoire of films for collective viewing by a group of colleagues;
- Purchase of a collection of fashion brands for sale in a clothing store;
- Choosing a site layout program, where experts rank the alternatives in what they believe to be the best to worst site layout program;
- Purchase of cars for the company - for ease of maintenance, etc.;
- Selection of a project for funding at the university research center;
- Organization of a corporate holiday and determination of the festive table menu;
- Providing a "cushion" of goods in the warehouse;
- Purchase of laptops for the office: for support and service, it is necessary that all devices are unified, so it is appropriate to choose one model among all;
- Preparation and approval of the organization's set of information security measures

Conclusions

Thus, the procedure of preliminary preferential voting for collectives consisting of several dozen members is proposed and studied. This task has been formalized and additional heuristics have been introduced to substantiate the approaches that will be used to determine the resulting ranking of job applicants.

The formulation of the problem of ordering a subset of alternatives is considered. Based on this examination, an algebraic approach to the analysis of incomplete rankings is proposed. A comparison of classical methods of group selection with the algorithm for determining the median with incomplete initial information was made. The scheme for solving the problem in the given formulation is described.

Conclusions

Thus, in this work, approaches to determining the resulting ranking based on incomplete expert rankings were investigated and the following main results were obtained:

- the concept of incomplete ranking was introduced;
- it is proposed to formulate the problem of determining the group ranking of alternatives based on incomplete expert rankings or multiple comparisons;
- approaches to aggregating expert data are considered, taking into account the features of incomplete information received from experts;
- developed algorithms for solving problems of calculating collective ranking based on individual multiple comparisons;
- computational experiments were conducted to study the described algorithms and features of ranking problems.