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Solving vector optimization problems on combinatorial configurations with fuzzily specified data

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1. Preliminaries

Let X is a universal set of alternatives, $\mu_A : X \to [0,1]$ is a given fuzzy subset of feasible alternatives, Y is a universal set of evaluations of the results of choices of alternatives from the set X, and $\mu_R : Y \times Y \to [0,1]$ is a given fuzzy preference ratio on the set.

Choices of alternatives are evaluated by fuzzy values of a given fuzzy objective function $\varphi : X \times Y \to [0,1]$. The task is to make a rational choice of alternatives based on the information given in the form described above.

Definition 1 [1, 2]. A fuzzy multiset $\tilde{X}$ defined on a universal multiset X is a set of pairs $\left(x, \mu_{\tilde{X}}(x)\right)$, where $x \in X$, $\mu_{\tilde{X}}(x)$ the function, $\mu_{\tilde{X}}(x)$: $X \to \left[0,1\right]$, is called the membership function of the multiset $\tilde{X}$.

The value $\mu_{\tilde{X}}(x)$ for a particular x is called the degree of belonging of this element to the fuzzy multiset $\tilde{X}$.

Let be a given fuzzy multiset

$\tilde{A}=\left\{\left(a_1,\mu_{\tilde{A}}(a_1)\right),\left(a_2,\mu_{\tilde{A}}(a_2)\right),...,\left(a_q,\mu_{\tilde{A}}(a_q)\right)\right\}$, its basis

$S(\tilde{A})=\left\{\left(e_1,\mu_{\tilde{A}}(e_1)\right),\left(e_2,\mu_{\tilde{A}}(e_2)\right),...,\left(e_k,\mu_{\tilde{A}}(e_k)\right)\right\}$, where

$\mu_{\tilde{A}}(e_i)=\min\left\{\mu_{\tilde{A}}\left(a_{i_j}\right)\middle| a_{i_j}=a_{i_t}, j\neq t,\forall i,j,t\in N_q\right\}$,

$e_j\in R_1\ \forall j\in N_k=\{1,...,k\}$ and multiplicity of elements

$k(e_j)=r_j, j\in N_k,\ \ r_1+r_2+...+r_k=q$.

An ordered fuzzy n -sample from a fuzzy multiset $\tilde{A}$ is called a set

$$a=\left(\left(a_{i_1},\mu_{\tilde{A}}(a_{i_2})\right),\left(a_{i_2},\mu_{\tilde{A}}(a_{i_2})\right),\ldots,a_{i_n},\mu_{\tilde{A}}(a_{i_n})\right), \tag{1}$$

where $a_{i_j}\in\tilde{A}\ \ \forall i_j\in N_k, \forall j\in N_k,\ i_s\neq i_t$, if $s\neq t\ \ \forall s\in N_k, \forall t\in N_k$.

Definition 2. [1] A fuzzy subset $P(\tilde{A})$ whose elements are fuzzy n-samples of the form (1) from a fuzzy multiset $\tilde{A}$ is called a fuzzy Euclidean combinatorial set if the following conditions are satisfied for an arbitrary pair of its elements

$$a=\left(a_1,\mu_{\tilde{A}}(a_1)\right),\left(a_2,\mu_{\tilde{A}}(a_2)\right),...,\left(a_n,\mu_{\tilde{A}}(a_n)\right) \text{ and}$$

$$b=\left(b_1,\mu_{\tilde{A}}(b_1)\right),\left(b_2,\mu_{\tilde{A}}(b_2)\right),...,\left(b_n,\mu_{\tilde{A}}(b_n)\right): \tag{2}$$

$$(a\neq b)\Leftrightarrow(\exists j\in N_n:\left(a_j,\mu_{\tilde{A}}(a_j)\right)\neq\left(b_j,\mu_{\tilde{A}}(b_j)\right),$$

that is, a set $P(\tilde{A})$ has the following property: two elements of a set $P(\tilde{A})$ are different from each other if, regardless of other differences, they differ in the order of placement of the symbols that make them up and in the degree of belonging to a fuzzy subset $P(\tilde{A})$.

Definition 3. [1] A convex combination of fuzzy sets $A_1, A_2, ..., A_n$ in R^n is called a fuzzy set A with a membership function of the form

$$\mu_A(x) = \sum_{i=1}^{n} \lambda_i \mu_i(x), \text{ where } \lambda_i \ge 0,\ i \in N_n, \sum_{i=1}^{n} \lambda_i = 1.$$

Along with the classical permutation polyhedron, we describe the general permutation polyhedron $\Pi_{nk}(\tilde{A})$, which is the convex hull of the general set of permutations $P_{nk}(\tilde{A})$ [1]:

$$\sum_{j=1}^{n} x_j \le \sum_{j=1}^{n} a_j,\ \sum_{j=1}^{i} x_{\alpha_j} \ge \sum_{j=1}^{i} a_j, \qquad (3)$$

$\alpha_j \in N_n,\ \ \alpha_j \ne \alpha_t,\ \forall j \ne t,\ \forall j,\ t \in N_i,\ \forall i \in N_n$, a $P_{nk}(A) = \text{vert} \prod_{nk}(A)$.

A fuzzy convex polyhedron can also be represented as a convex hull of a fuzzy combinatorial set of permutations: $\Pi_{nk}(\tilde{A}) = \text{conv}\, P_{nk}(\tilde{A})$.

2.Formulation of the vector optimization problem on the combinatorial configuration of permutations with fuzzy specified data

The vector problem of combinatorial optimization is considered

$$Z(F,X):\max\left\{F(x)\mid x\in X\subset R^n\right\},$$

$$F(x)=(f_1(x),...,f_l(x)),$$

$$f_i:R^n\to R,\ i\in N_l,$$

$$X=\operatorname{vert}\Pi_{nk}(A)\cap D\neq\varnothing,\ \Pi_{nk}(A)=\operatorname{conv}P_{nk}(A),$$

where $P_{nk}(A)$ − combinatorial set of permutations, $D\subset R^n$ − convex polyhedron. A fuzzy subset $\tilde{X}=\left\{x,\mu_{\tilde{X}}(x)\right\}$, is given on the set X

$\tilde{X}=\left\{x,\mu_{\tilde{X}}(x)\right\}$, where $x\in X$, and $\mu_{\tilde{X}}(x):X\to[0,1]$ − set membership function $\tilde{X}$.

A fuzzy decision making problem defined over a feasible set X of decision variable vectors assumes the existence of several fuzzy goals $G_k, k = 1,\ldots,l,$ that are fuzzy subsets of X under a set of fuzzy restrictions $R_i, i = 1,\ldots,m,$ that are also fuzzy subsets of X. Bellman and Zadeh [1] described a solution to such problem (i.e. a decision), through a fuzzy subset of X, i.e. a set $\{x, \mu D(x) \mid x \in X\},$ where the membership function

$\mu_D : X \to [0,1]$ is defined by aggregating the fuzzy goals and restrictions using the min operator

$$\mu_D(x) = \min\left(\left\{\mu_{G_k}(x) \mid k = 1,\ldots,l\right\} \cup \left\{\mu_{R_i}(x) \mid i = 1,\ldots,m\right\}\right).$$

The classic way to construct a fuzzy goal related to any kind of objective functions f_i, that has to be maximized is to involve a threshold (g_i) and a tolerated limit $(t_i < g_i)$ on the given threshold, and define the membership function $\mu_{f_i}(f_i(x))\,\mu_{\tilde{X}}(f_i(x))$, where

$$\mu_{f_i}(f_i(x)) = \begin{cases} 0, \ f_i(x) < t_i, \\ 1 - \dfrac{f_i(x) - t_i}{g_i - t_i}, t_i \le f_i(x) \le g_i, \ i \in N_l = \{1,...,l\}. \\ 1, \ f_i(x) > g_i. \end{cases}$$

Further on, Zimmermann [4] proposed the following mathematical problem

$$\max \alpha$$

$$\mu_{G_k}(x) \ge \alpha, \ k = 1,...,l, \ \mu_{R_i}(x) \ge 0, \ i = 1,...,m,$$

$0 \le \alpha \le 1, \ x \in X$, to derive the optimal decision, namely the solution with the maximal membership value.

3. Approaches to solving the vector optimization problem on combinatorial configurations with fuzzy specified data

3.1 Problem solving based on the guaranteed result method

The following optimization problem:

$$z = \min_{i=1,2,\ldots l} f_i(x) \to \max, \ x \in X.$$

For normalized criteria

$$\lambda_k(x) = \frac{f_k(x)}{f_k^*} : R^n \to R, \ k \in N_l,$$

where $f_k^* = \max_{x \in X} f_k(x) : R^n \to R, \ k \in N_l,$ the maximin problem is formulated in the form:

$$z = \min_{k \in N_l} \lambda_k(x) \to \max : x \in X, R^n \to R, \ k \in N_l. \quad (4)$$

Problem (4) is equivalent to problem

$$z=\lambda\to\max \tag{5}$$

under conditions

$$\begin{cases}\lambda \le \lambda_k(x), k \in N_l,\\ x \in X,\end{cases} \tag{6}$$

$$X = \operatorname{vert}\Pi_{nk}(A)\bigcap D \ne \varnothing,\ \Pi_{nk}(A) = \operatorname{conv} P_{nk}(A),$$

where $P_{nk}(A)$ − combinatorial set of permutations, $D \subset R^n$ − convex polyhedral set.

Problem (5) − (6) is called a λ-problem. It has a linear objective function and $m+l$ constraints.

Solving the problem of multicriteria optimization by the method of a guaranteed result, as a rule, goes through the following stages:

1. Development of a mathematical model of the system based on set goals and limitations; at the same time, the opinion of experts is often used.
2. Preliminary analysis of the system separately for each partial criterion; use methods and software tools of single-criteria optimization.
3. Standardization of criteria.
4. Solving the multicriteria optimization problem with equivalent criteria.
5. Determining the priorities of the criteria and solving the multi-criteria optimization problem with assigned priorities.

3.2 Approach to solving the problem based on the method of successive concessions

Let x and y be two fuzzy numbers with carriers $S_x = (a_1, a_2)$ and $S_y = (a_1, a_2)$, respectively:

$$a_2 > a_1, b_2 > b_1 \ ;$$

$$g : R^1 \times R^1 \to R^1 \text{ - some function.}$$

Then, according to the principle of generalization, the fuzzy number is determined by the membership function

$$\mu_D(z) = \sup_{\substack{g(a,b)=z \\ a\in S_x,\ \ b\in S_y}} \min\left\{\mu_x(a), \mu_y(b)\right\} \qquad (7)$$

Let us denote $\otimes -$ one of the four arithmetic operations: $+, -, \cdot, /$; $g(a,b) = a \otimes b$.

Then formula (7) determines the result of an arithmetic operation $\otimes$ on fuzzy numbers x and y .

If $g(\cdot)-$ is a function of not two, but n arguments, then the principle of generalization is formulated analogously to formula (7).

Definition 4. Two fuzzy values (two numbers) $\left(x_1, \mu_1(x_1)\right)$ and $\left(x_2, \mu_2(x_2)\right)$ we will consider equal if $x_1 = x_2$ i $\mu_1(x_1) = \mu_2(x_2)$.

Definition 5. If the condition $x_1 \geq x_2$, $\mu_1(x_1) \geq \mu_2(x_2)$ and one of these inequalities is strict, then the fuzzy quantity $\left(x_1, \mu_1(x_1)\right)$ is greater than the fuzzy quantity $\left(x_2, \mu_2(x_2)\right)$.

An approach based on the method of successive concessions has been developed. When solving a multi-criteria problem by the method of successive concessions, a qualitative analysis of the relative importance of partial criteria is first made. The peculiarity of this method is that the problem criteria must be pre-numbered in descending order of their importance, thus the main criterion $f_1(x)$ is less important than $f_2(x)$, followed by other partial criteria $f_3(x)$, $f_4(x)$,…, $f_l(x)$. The most important criterion is maximized $f_1(x)$ and its largest value is determined f_1^*.

Then the value of the permissible reduction (concession) $\Delta_1 \geq 0$ of the criterion $f_1(x)$ is assigned and the largest value f_2^* of the second criterion is found $f_2(x)$, provided that the value of the first criterion must not be less than $f_1^* - \Delta_1$.

The amount of the concession is again assigned $\Delta_2 \geq 0$, but according to the second criterion, which is used together with the first when finding the conditional maximum of the third criterion, etc. Finally, the last most important criterion is maximized $f_l(x)$, provided that the value of each criterion $f_r(x)$ from $l-1$ the previous ones must be no less than the corresponding value $f_r^* - \Delta_r$, then the solutions obtained as a result are considered optimal.

Thus, the choice of the solution of the problem is carried out by performing a multi-step procedure and consists in sequentially including the constraints of the problem $Z(F, X)$ and taking into account the structural features of its admissible area.

The optimal solution is considered to be the solution of the last problem from the following sequence of problems:

$$f_1^* = \max\left\{ f_1(x) \middle| x \in X \right\},$$

$$f_2^* = \max\left\{ f_2(x) \middle| x \in X,\ f_1(x) \ge f_1^* - \Delta_1 \right\}, \ldots,$$

$$f_l^* = \max\left\{ f_l(x) \middle| x \in X,\ f_{r-1}(x) \ge f_{r-1}^* - \Delta_{r-1}, r \in N_l \right\}.$$

It should be noted that in the case when all Δ_r are zero, the method of successive concessions selects only lexicographically optimal strategies; these strategies deliver the largest solution to the most important criterion in the set of admissible values $f_1(x)$.

Conclusion

The paper presents the formulation of the vector optimization problem on the combinatorial configuration of permutations with vaguely specified data. The vagueness is specified in the description of the objective function and the admissible domain of the problem.

The Edgeworth–Pareto principle applies to a class of multicriteria problems in which the set of possible solutions is fuzzy, or the objective function has fuzzy parameters.

Methods of solving multi-criteria problems with vague input information are presented. Depending on the specifics of the task, it is possible to apply other methods of multicriteria selection modified in case of vaguely specified information.

The generalization of clear methods, as a rule, does not present particular difficulties, if the methods of presenting vague concepts, implementing vague calculations, comparing vague numbers, and forming a vague set of better alternatives are chosen in accordance with the conditions of the problem being solved.

As a result of the research of the vector combinatorial problem, which is based on the use of information about the convex hull of the admissible domain, the study of the properties of the polyhedron, the vertices of which are defined by a vaguely specified combinatorial set of permutations, a method of solving complex multicriteria problems on the specified combinatorial set was developed and substantiated.

References

[1] Semenova N.V., Kolechkina L.N. and Nagirna A.M. Vector optimization problems with linear criteria over a fuzzy combinatorial set of alternatives. Cybernetics and Systems Analysis. 2011. Vol. 47, N 2. P. 250–259. https://doi.org/10.1007/s10559-011-9307-5.

[2] Zheldak T.A., Koriashkina L.S., Us S.A. Fuzzy sets in management and decision-making systems. Ministry of Education and Science of Ukraine, National technical Dniprovska University polytechnic". Dnipro: NTU "DP". 2020. 387 p. http://ir.nmu.org.ua/handle/123456789/156356.

[3] Ehrgott M. Multicriteria optimization. Berlin; Heidelberg: Springer, 2005. 323 p.

[4] Johannes J. Vector optimization. Theory, applications, and extensions. Second edition. Berlin; Heidelberg: Springer-Verlag, 2011. 481 p.

[5] Hulianytskyi L., Riasna I. Formalization and classification of combinatorial optimization problems. Optimization Methods and Applications, S. Butenko et al. (eds.). New York: Springer, 2017. P. 239–250.

Thank you for your attention!