Fractal step multiwavelets and multiwavelet packets - a new multiwavelet technology for image processing and coding

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ADVANTAGES OF USING KNOWN MULTIWAVELETS OVER WAVELETS

For the first time, multiwavelets were obtained by Geronimo et al. (Geronimo, Hardin, Massupust) in 1994.

Martin and Bell (2001) constructed new multi-wavelets and multi-wavelet packets (MWPs) and experimentally showed that in image coding, multi-wavelet packets improve the compression performance of texture-rich images compared to the known best wavelets. But multiwavelet packets increase the computational complexity as a result of the basis selection.

Ashok and Reddy (2011) showed that using SA4 multiwavelet for medical image compression provides a 3 dB improvement in PSNR estimation. Rema et al. [2017, 2018] showed that multiwavelets in fingerprint compression improve PSNR by 2.52 dB and 4.23 dB, respectively, for two databases at compression ratios of 100:1 and 80:1. Sindhuja and others. (2012) showed that multiwavelets are an order or two of magnitude better at detecting and transmitting contours.

Fractal step functions, fractal multiwavelets and multiwavelet packets

In 2015, Hnativ introduced a new class of normalized fractal step functions (FSF), which are self-similar and have a fractal dimension. Based on them, a method of construction for a whole family of orthonormal basis systems of a new class of fractal multiwavelets of different shapes with linear and nonlinear changes of values has been developed. A new class of fractal stepped multiwavelets (FSMW) based on FSF was constructed and their transforms were developed with fast algorithms of linear computational complexity (2017). Estimates have been obtained of the computational complexity of a fast algorithm for computing a discrete multiwavelet transform with multiwavelet packets of size $N \text{x} N$, which requires $M_N=3N/2-4$ multiplications and $A_N=17N/4-6$ additions.

FSMWs are symmetrical and orthogonal and have high frequency localization, which improves the presentation of high-frequency signals and images. Multiwavelet packet of size $N \text{x} N$ contains N-1 multiwavelets, which is many more than the known classical multiwavelets, so they have higher accuracy representing high-frequency signals and images with a lot of texture.

In 2021, orthonormal bases of fractal stepped multiwavelets and multiwavelet packets were described, and based on them, a method and algorithm of fast multiwavelet transform of low computational complexity was developed, which has multiplicative complexity that is 70 times lower and additive complexity that is 20 times lower compared to the well-known classical Mallat algorithm.

The discrete multiwavelet transform

For a function $f(t)$ represented by a sequence of numbers, the discrete multiwavelet transform (DMWT) is defined by a pair of discrete wavelet transforms [21]

$$W_{\varphi}(j_0,i_0)=\frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}f(t)\varphi_{(j_0,i_0)}(t), j\geq j_0, \quad (1)$$

$$W_{\psi_i^{(k)}}(j,i)=\frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}f(t)\psi_{j,i}^{(k)}(t), i=\overline{1,N-1}, \quad (2)$$

where N - number of multiwavelet functions of rank k that represent a multiwavelet packet of size $N\times N$, $N=2^p, p\geq 2$, $\psi_{j,i}^{(k)}(t)$ - i-th function of a fractal step multiwavelet of rank k [20], $k=0,1,\ldots p-2$ for j-th level of decomposition, $j=0,1,2,\ldots m$; $\varphi_{j_0,i_0}(t)$ - Haar scaling function $j_0=0$ and $i_0=0$, $\varphi_{0,0}(t)=1$.

In this case, addition is performed for the values of t, i, and j.

For function $f(x), x=\overline{0,2^{n-1}}, m=\left[\frac{n}{p}\right]$.

Fast multiwavelet transform

Hnativ, 2021_a fast multiwavelet transform (FMWT) is proposed, which is an efficient method for computing the DMWT. It utilizes the interdependencies between the coefficients of the DMWT at neighboring levels of decomposition. Approximation coefficients $W_{\varphi}\left(j+1,i_0\right)$ and detail coefficients $W_{\psi^{(k)}}\left(j+1,i\right)$ level $j+1$ can be calculated through approximation coefficients $W_{\varphi}\left(j,i_0\right)$ level j.

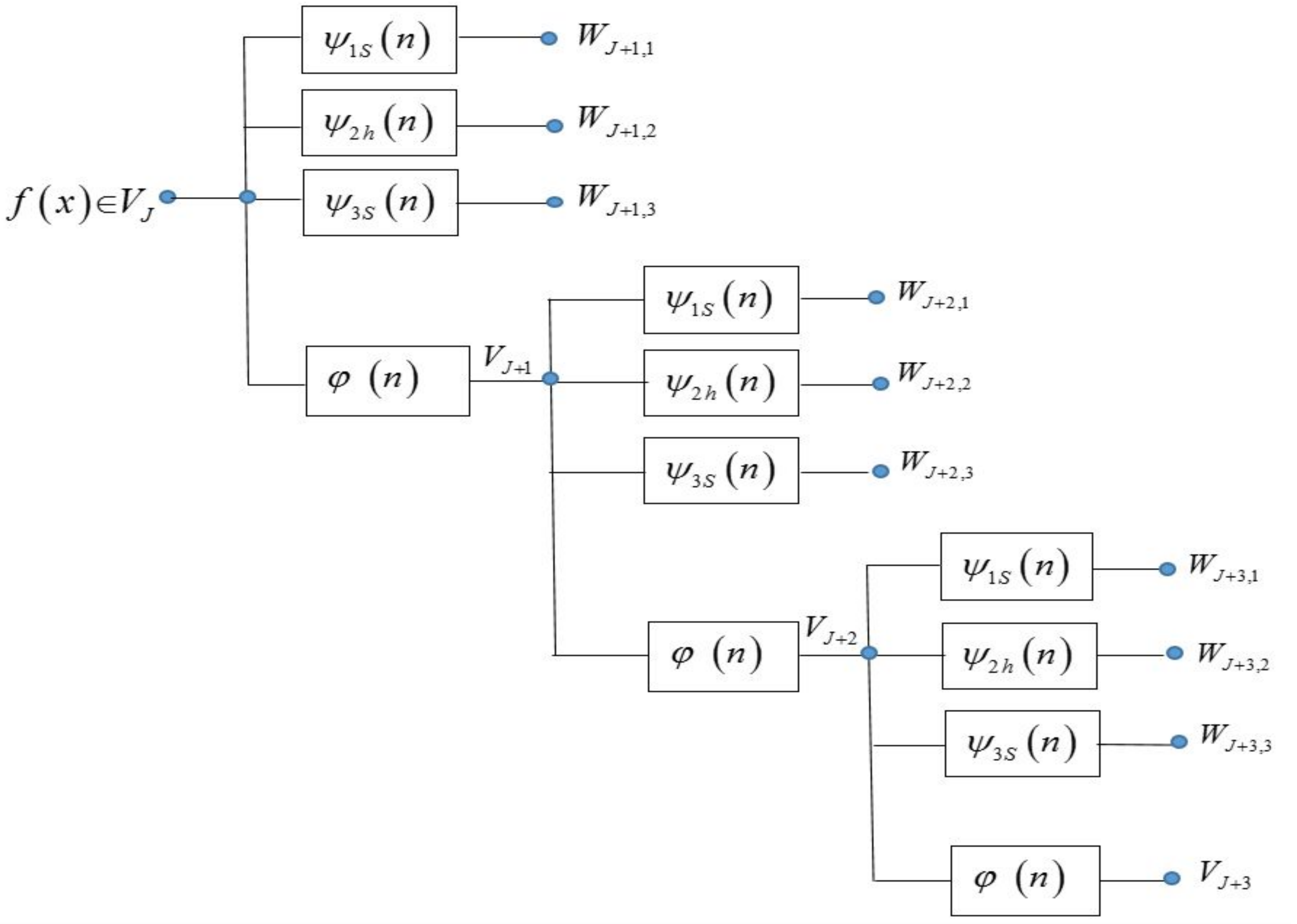
Theorem (Hnativ, 2021)

$$W_{\varphi}\left(j+1,i_0\right)=\sum_{n}\varphi_{i_0}\left(n\right)W_{\varphi}\left(j,n\right),\ n=0,1,\ldots N-1, \quad (3)$$

$$W_{\psi_i^{(k)}}\left(j,i\right)=\frac{1}{\sqrt{N_1}}\sum_{t=0}^{N_1-1}f\left(t\right)\psi_{j,i}^{(k)}\left(t\right),\ i=\overline{1,N_1-1},\ i=1,\ldots N-1,\ k=0,1,\ldots p-2. \quad (4)$$

Expressions (3) and (4) represent the algorithm of fast multiwavelet transform, which can be computed using only scalar product operations without convolution (equivalent to filtering) and downsampling by a factor of 2, as required by the well-known Mallat algorithm for fast wavelet transform.

Block diagram of a three-level fast multiwavelet transform with multiwavelet packets of size 4x4



A method for constructing a discrete multiwavelet transform based on a 8x8 mu

Consider the matrix SWT_8^* DMWT, which represents MWT order 8x8 with rearranged lines on the base **binary inverse permutations**

$$SWT_8^* = P_8 SWT_8, \tag{5}$$

where P_8 - matrix 8x8 **binary inverse permutations,** $P_8(0,7) = (0,4,2,6,1,5,3,7)$.

Matrix SWT_8^* order 8x8 DMWT with permuted rows can be written through a matrix MWT

$$SWT_8^* = B_8 S_8^*, \tag{6}$$

where S_8^* - matrix 8x8 MWT, B_8 - diagonal 8x8 matrix of normalization coefficients.

Matrix S_8^* can be built based on the recursive method:

$$S_8^* = \tilde{B}_8 R_8 H_8 diag\left[S_4^*, S_4^*\right], \tag{7}$$

where H_8 - factor matrix 8x8 with nonzero elements ± 1, $R_8 = diag\left[I_4, R_3^{(1)}, 1\right]$, $R_3^{(1)} = \begin{bmatrix} r_1^{(1)} & & s_1^{(1)} \\ & 1 & \\ -s_2^{(1)} & & r_2^{(1)} \end{bmatrix}$,

where $R_3^{(1)}$ - order 3x3 first rank rotation-compression/extension operator $(k=1)$ with constants $r_1^{(1)}$, $r_2^{(1)}$, $s_1^{(1)}$ i $s_2^{(1)}$, which satisfy the condition: $r_1^{(1)} + s_1^{(1)} = r_2^{(1)} - s_2^{(1)} = 1$ **(A)**

$r_1^2 + s_1^2 < 1$, **(rotation-compression),** $r_2^2 + s_2^2 > 1$ **- rotation - extension. (B)**

This is a new operator introduced by Hnativ (2021) and generalizes the well-known classical Givens rotation operator that satisfies the condition $r^2+s^2=1$.

Matrix $\mathbf{S}_8$ MWT size 8x8

$$S_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & s_1 & s_2 & s_3 & -s_3 & -s_2 & -s_1 & -1 \\ 1 & s_4 & -s_4 & -1 & -1 & -s_4 & s_4 & 1 \\ 1 & q_1 & -q_2 & -q_3 & q_3 & q_2 & -q_1 & -1 \\ \sqrt{2}(1 & -1 & -1 & 1 & & & &) \\ \sqrt{2}(& & & & 1 & -1 & -1 & 1) \\ \sqrt{2}(1 & -q_4 & q_4 & -1 & & & &) \\ \sqrt{2}(& & & & 1 & -q_4 & q_4 & -1) \end{bmatrix} \qquad (16)$$

For example, elements s_i, q_i, $i=\overline{1,4}$, matrix S_8 for functions $\psi_1^{(1)}(t)$, $\psi_3^{(1)}(t)$, $\psi_2^{(0)}(t)$, $\psi_{3(1,j)}^{(0)}(2t-j)$, $j=0{,}1$ with non-linear FSF [19,20,21] acquire such values: $s_1=7/8$, $s_2=3/8$, $s_3=1/4$, $s_4=2/3$, $q_1=7/17$, $q_2=33/17$, $q_3=43/17$, $q_4=3/2$. At the same time, the constants of the matrix $R_3^{(1)}$ take the following values: $r_{1,H}^{(1)}=5/8$, $r_{2,H}^{(1)}=30/17$, $s_{1,H}^{(1)}=3/8$ i $s_{2,H}^{(1)}=13/17$.

Algorithm for fast calculation of 8-point DMWT

Based on the recurrent matrix representation of the multiwavelet packet size $N \times N$ in [Hnativ, 2017] the factorized representation of the matrix as a product $2\log_2 N - 1$ matrix is obtained. This makes it possible to build a fast calculation algorithm (FA) DMWT. Thus, the matrix SWT_8^* can be represented as a product of five factor matrices:

$$SWT_8^* = B_8 S_{8,5} S_{8,4} S_{8,3} S_{8,2} S_{8,1} \qquad (22)$$

where $S_{8,k}$ - k-i, $k = \overline{1,5}$, factor-matrix 8x8 of the algorithm proposed in [Hnativ 2017] for fast calculation of 8-point DMWT, B_8 - diagonal 8x8 matrix of normalization coefficients:

$$S_{8,1} = I_4 \otimes H_2,\ S_{8,2} = diag\left[H_4, H_4\right],$$

$$S_{8,3} = diag\left[I_2, R_2^{(0)}, I_2, R_2^{(0)}\right],$$

$$S_{8,4} = H_8,\ S_{8,5} = R_8 = diag\left[I_4, R_3^{(1)}, 1\right], \qquad (23)$$

$$B_8 = 2^{-3/2} diag\left[1, \sqrt{2}, b_2, b_4, b_1, \sqrt{2}, b_3, b_4\right].\ b_1 = 8\sqrt{2/63},\ b_2 = 3\sqrt{2/13},\ b_3 = 17/\sqrt{819},\ b_4 = 4/\sqrt{13}$$

Two-dimensional discrete multiwavelet transform

Let's define DMWT functions $f(x,y)$ sizes $M \times N$ as follows:

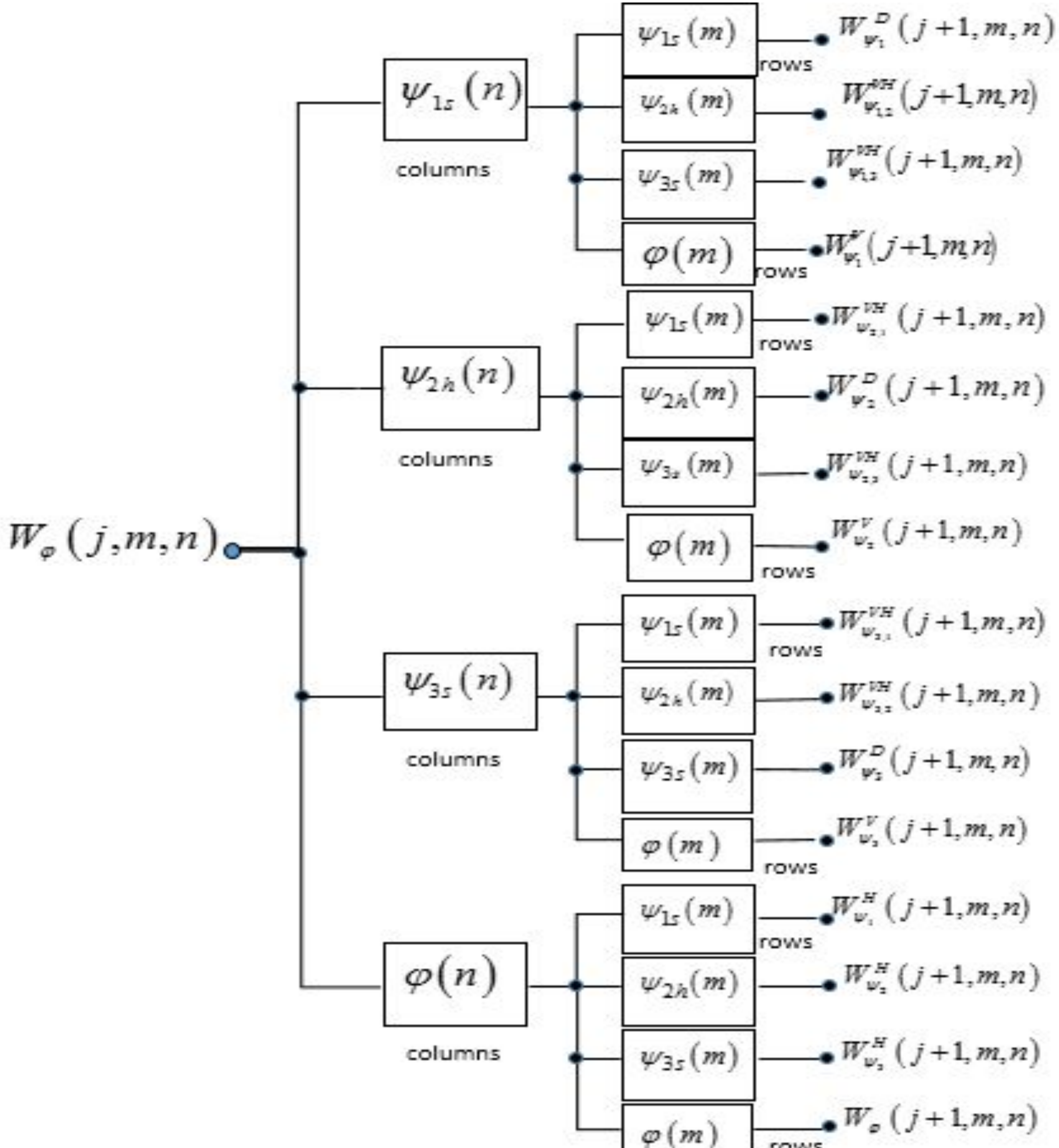
$$W_{\varphi}(j_0,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\varphi_{j_0}(x,y) \quad (27)$$

$$W_{\psi_i}^{i'}(j,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\psi_{i,j,m,n}^{i'}(x,y),\ i'=\{H,V,D\} \quad i=\overline{1,N-1} \quad (28)$$

As in the one-dimensional case, j_0 - the initial level of the schedule, and coefficients $W_{\varphi}(j_0,m,n)$ determine the approximation of the function $f(x,y)$ at level j_0. Coefficients $W_{\psi_k}^{i}(j,x,y)$ define horizontal, vertical and diagonal details for levels $j \geq j_0$. We consider $j_0=1$ and choose numbers N and M so that they are a power of two. $N=M=2^{J'}$, $J' \geq 2$, $m,n=0,1,2,...2^{j'}-1$.

With, $J=0,1,2,...r$, $r=[j'/p]$, $N_1=2^p$, $p \geq 2$.

Block diagram of a one-level two-dimensional fast multiwavelet transform (FMWT) with one multiwavelet packet of size 4x4



Two-dimensional FMWT for one decomposition level

$W_{\varphi}(j,m,n)$ →

$W_{\varphi}(j+1)$	$W_{\psi_1}^{H}(j+1)$	$W_{\psi_2}^{H}(j+1)$	$W_{\psi_3}^{H}(j+1)$
$W_{\psi_1}^{V}(j+1)$	$W_{\psi_1}^{D}(j+1)$	$W_{\psi_{1,2}}^{VH}(j+1)$	$W_{\psi_{1,3}}^{VH}(j+1)$
$W_{\psi_2}^{V}(j+1)$	$W_{\psi_{2,1}}^{VH}(j+1)$	$W_{\psi_2}^{D}(j+1)$	$W_{\psi_{2,3}}^{VH}(j+1)$
$W_{\psi_3}^{V}(j+1)$	$W_{\psi_{3,1}}^{VH}(j+1)$	$W_{\psi_{3,2}}^{VH}(j+1)$	$W_{\psi_3}^{D}(j+1)$

W_{ϕ}, $\left(W_{\psi_1}^{H}, W_{\psi_2}^{H}, W_{\psi_3}^{H}\right)$, $\left(W_{\psi_1}^{V}, W_{\psi_2}^{V}, W_{\psi_3}^{V}\right)$, $\left(W_{\psi_{1,2}}^{VH}, W_{\psi_{1,3}}^{VH}, W_{\psi_{2,1}}^{VH}, W_{\psi_{2,3}}^{VH}, W_{\psi_{3,1}}^{VH}, W_{\psi_{3,2}}^{VH}\right)$ i $\left(W_{\psi_1}^{D}, W_{\psi_2}^{D}, W_{\psi_3}^{D}\right)$.

A method of image coding based on multiwavelets and multiwavelet packets using 2D FMWT

As shown by [Gonzalez D., Woods R. 2002], when the number of expansion levels increases beyond three, the number of coefficients that are set to zero changes little.

The paper proposes a new multi-wavelet image coding method based on a three-level two-dimensional **FMWT** with multi-wavelet packets of given sizes, as a new multi-wavelet technology for image coding.

Computational complexity of the image coding method based on the 8-point 2D FMWT for three levels of decomposition

For a three-level schedule scheme when encoding an image of size *N*x*N* based on an 8-point 2D FMWT with an 8x8 multiwavelet packet, it is necessary $M_{N,8} = M_8 \sum_{i=1}^{3} 2N^2 / 8^{2i-1} = \frac{M_8 4161N^2}{16384}$ of multiplications that at M_8=8 makes up $\frac{4161N^2}{2048}$ multiplications, or by one pixel is required $M_{8/p} = 2,03$ **multiplication by pixel**, and $A_{N,8} = A_8 \sum_{i=1}^{3} 2N^2 / 8^{2i-1} = \frac{A_8 4161N^2}{16384}$ additions that at A_8 =28 makes up $\frac{7*4161N^2}{4096}$ additions, by one pixel is required $A_{8/p} = 7,11$ **additions by one pixel** for functions with linear changes. For functions with non-linear changes - A'_8=32 is required $A'_{N,8} == \frac{32*4161N^2}{16384} = \frac{4161N^2}{512}$ additions, that makes up **8,13 additions by one pixel.**

Comparative analysis of the computational complexity of the proposed method with the well-known classical Mallat method of image coding

The well-known Mallat algorithm 2D FWT for filters with K non-zero coefficients of the whole-tree wavelet packet of $\log_2 N$ depth for an NxN image requires $2K N^2 \log_2 N$ of multiplications and additions, which at K=8 makes $16N^2 \log_2 N$ operations, or 16 $\log_2 N$ operations per pixel.

The proposed image coding method based on the three-level 8-point 2D FMWT compared to the well-known Mallat algorithm 2D FWT for filters with 8 non-zero coefficients requires $K_M = \frac{16\log_2 N}{2,03} = 7,88\log_2 N$ times fewer multiplications that for $N=2^{10}$ makes up **78,8 times** and in $K_A = \frac{16\log_2 N}{7,11} = 2,25\log_2 N$ times less additions, which is **22.5 times** less for functions with **linear changes in values**. For functions with **non-linear changes in values** $K_{A'} = \frac{16\log_2 N}{8,13} = 1,97\log_2 N$ times less additions, which is **19.7 times** less. The paper proposes a multi-wavelet method of image coding based on a three-level two-dimensional FMWT with a multi-wavelet packet of size 8x8. The proposed method of image coding based on a three-level 8-point two-dimensional FMWT compared to the well-known classical Mallat algorithm FWT for filters with 8 non-zero coefficients has 7,88 $\log_2 N$ times the lower multiplicative complexity, which for $N=2^{10}$ is 78.8 times and needs in $2,25\log_2 N$ times less additions, which is 22.5 times less for functions with linear changes.

Experimental results and their analysis

A new computer-based image coding technology based on 2D **FMWT** with multi-wavelet packets of a given size for three levels of scheduling has been developed. For seven test images of classes A, B, C with a resolution of 2560x1536, 2048x1280 and 1280x768. Experimental results showed that the proposed method of multiwavelet coding, in comparison with the known block method based on the integer cosine transform of order 32, which is used in the H.265 video coding standard, provides better visual quality, since there are no block distortions, which are amplified at high degrees of compression for H.265. At the same time, the reduction of computational complexity by multiplication is 11 times.

Conclusions

The construction methods and algorithms of two-dimensional (2D) discrete fractal step multiwavelets and multiwavelet packets, 2D discrete multiwavelet transforms with multiwavelet packets of given sizes for different levels of the schedule without performing convolution and sample thinning operations, unlike the classical Mallat method, have been developed. Algorithms of 2D fast multiwavelet transforms have been developed based on fast algorithms for calculating discrete multiwavelet transforms with multiwavelet packets of given sizes of linear computational complexity for different levels of the decomposition of low computational complexity for more accurate and faster image analysis and coding. A method and algorithms for image coding based on 2D FMWT are proposed as a new multi-wavelet technology for image coding. If based on a three-level 8-point 2D FMWT it has 78.8 times lower multiplicative complexity and 22.5 times lower additive complexity compared to the classic Mallat algorithm that is 2D FWT for filters with 8 non-zero coefficients.

Thank you for your attention!