Convergence of adaptive forward-reflected-backward algorithm for solving variational inequalities

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New methods of research of correctness and solution search for discrete optimization problems, variational inequalities and their applications,
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Variational Inequalities and Optimization

- E – real vector space with norm $\|\cdot\|$
- E^* – dual of E, with dual norm $\|\cdot\|_*$
- $\langle f, x \rangle$ – value of $f \in E^*$ at $x \in E$

Variational Inequality

$$\text{find} \quad x \in C \quad \text{such that} \quad \langle Ax, y - x \rangle \geq 0 \quad \forall\, y \in C \qquad (VI)$$

Smooth constrained optimization

Assumption: f – convex smooth function, X – closed convex set

$$f(x^*) = \min_{x \in X} f(x) \iff \langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in X$$

Variational Inequalities and Optimization

Saddle point problem

Assumption: f – convex-concave and smooth, X, Y – closed and convex

$$f(x^*, y^*) = \min_{x \in X} \max_{y \in Y} f(x, y)$$

$$\Updownarrow$$

$$\left\langle \underbrace{\begin{pmatrix} \nabla_x f(x^*, y^*) \\ -\nabla_y f(x^*, y^*) \end{pmatrix}}_{Az^*}, \underbrace{\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}}_{z - z^*} \right\rangle \geq 0 \quad \forall \underbrace{(x, y)}_{z} \in \underbrace{X \times Y}_{Z}$$

Variational Inequality

$$\text{find} \quad x \in Z \quad \text{such that} \quad \langle Az^*, z - z^* \rangle \geq 0 \quad \forall z \in Z$$

Assumption: A is monotone: $\langle Az_1 - Az_2, z_1 - z_2 \rangle \geq 0 \quad \forall z_1, z_2 \in Z$

Variational Inequalities and Optimization

1. Matrix games:

$$\min_{x \in \Delta^n} \max_{y \in \Delta^m} \langle Px, y \rangle$$

2. Constrained optimization:

$$\min_x f(x) \;\; s.t. \;\; g(x) \leq 0 \quad \longrightarrow \quad \min_x \max_{y \geq 0} \left(f(x) + yg(x) \right)$$

3. Structural minimization:

$$f(x) + g(Ax) \to \min_x \quad \longrightarrow \quad \min_x \max_y \left(f(x) + \langle Ax, y \rangle - g^*(y) \right)$$

4. Structural minimization (discrete maximum function):

$$\max_{i=1,\ldots,m} f_i(x) \to \min_x \quad \longrightarrow \quad \min_x \max_{y \in \Delta^m} \sum_{i=1}^{m} y_i f_i(x)$$

5. Machine Learning:

- Adversarial training: $\min_x \sum_i L(x, a_i, b_i) \;\to\; \min_x \max_{\omega \in \Omega} \sum_i L(x, a_i + \omega, b_i)$
- Generative adversarial networks (GANs)

Some geometry of Banach spaces

Convexity:

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \quad x, y \in S_E, \quad \|x-y\| = \varepsilon\right\} \quad \forall \varepsilon \in (0,2]$$

is called modulus of convexity of E

- E is uniformly convex: $\delta_E(\varepsilon) > 0 \ \forall \varepsilon \in (0,2]$
- E is 2-uniformly convex: $\exists c > 0 : \delta_E(\varepsilon) \geq c\varepsilon^2 \ \ \forall \varepsilon \in (0,2]$

Smoothness:

- E is smooth, if:

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

  exists for all x, $y \in S_E$.
- E is uniformly smooth, if the limit is attained uniformly for x, $y \in S_E$.

L_p $(1 < p \leq 2)$ are 2-uniformly convex and uniformly smooth

Some geometry of Banach spaces

Normalized duality mapping:

$$J : E \to 2^{E^*} : \quad Jx = \left\{ x^* \in E^* : \ \langle x^*, x \rangle = \|x\|^2 = \|x^*\|_*^2 \right\}.$$

If E is smooth, then J is single-valued $\left(J = \nabla \frac{1}{2} \|\cdot\|^2\right)$.

Examples:

- $Jx = \|x\|_{L_p}^{2-p} |x|^{p-2} x \in L_q, \ \ x \in L_p$
- $Jx = \|x\|_{W_p^m}^{2-p} \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha (|D^\alpha x|^{p-2} D^\alpha x) \in W_q^{-m}, \ \ x \in W_p^m$

Alber generalized projection

Let E be a smooth Banach space. Consider the Alber functional

$$\phi(x,y) = \|x\|^2 - 2\langle Jy, x\rangle + \|y\|^2 \quad \forall x, y \in E.$$

Let K be a non-empty closed and convex subset of a reflexive, strictly convex and smooth space E. It is known that for each $x \in E$ there is a unique point $z \in K$ such that

$$\phi(z,x) = \inf_{y \in K} \phi(y,x).$$

This point z is denoted by $\Pi_K x$, and the corresponding operator $\Pi_K : E \to K$ is called the generalized projection of E onto K.

Let K be a closed and convex subset of a reflexive, strictly convex and smooth space E, $x \in E$, $z \in K$. Then

$$z = \Pi_K x \quad \Leftrightarrow \quad \langle Jz - Jx, y - z\rangle \geq 0 \quad \forall y \in K \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y,x) \quad \forall y \in K.$$

Aoyama–Kohsaka inequality

Let E be a 2-uniformly convex and smooth Banach space. Then, for some $\mu \geq 1$, the inequality holds

$$\phi(x, y) \geq \frac{1}{\mu} \|x - y\|^2 \quad \forall x, y \in E.$$

Examples:

L_p, W_p^m $(1 < p \leq 2)$: $\mu = \frac{1}{p-1}$

Variational Inequality

$$\text{find } x \in C \text{ such that } \quad \langle Ax, y - x \rangle \geq 0 \quad \forall y \in C \tag{1}$$

where

- E – 2-uniformly convex and uniformly smooth
- $C \subseteq E$
- $A : E \to E^*$

Let denote the solution set of the problem (1) by S.

Assumption:

- $C \subseteq E$ – convex and closed
- operator $A : E \to E^*$ – monotone and Lipschitz continuous on C, $L > 0$
- $S \neq \emptyset$

Fixed point formulation:

$$x = \Pi_C J^{-1} \left(Jx - \lambda Ax \right), \tag{2}$$

where $\lambda > 0$.

State of the Art

H. Iiduka, W. Takahashi, 2008 ($A : E \to E^*$ is inverse-strongly-monotone)

$$x_{n+1} = \Pi_C J^{-1} \left(J x_n - \lambda A x_n \right).$$

Y. Shehu, 2020 ($A : E \to E^*$ is monotone Lipschitz continuous operator)

$$\begin{aligned} y_n &= \Pi_C J^{-1} \left(J x_n - \lambda A x_n \right), \\ x_{n+1} &= J^{-1} \left(J y_n - \lambda \left(A y_n - A x_n \right) \right). \end{aligned}$$

Forward-Reflected-Backward (Alber projection setting)

Algorithm 1

Init. Choose $x_0 \in E$, $x_1 \in E$, $\lambda \in \left(0, \frac{1}{2\mu L}\right)$. Let $n = 1$.

1. Calculate

$$x_{n+1} = \Pi_C J^{-1}\left(Jx_n - \lambda Ax_n - \underline{\lambda\left(Ax_n - Ax_{n-1}\right)}\right).$$

2. If $x_{n-1} = x_n = x_{n+1}$, then STOP, else let $n := n + 1$ and go to 1.

In Hilbert space

Yu. Malitsky, M. K. Tam, 2020

$$x_{n+1} = P_C\left(x_n - 2\lambda Ax_n + \lambda Ax_{n-1}\right)$$

Yu. Malitsky, 2015

$$x_{n+1} = P_C\left(x_n - \lambda A(2x_n - x_{n-1})\right)$$

Analysis 1

Lemma 1. The following inequality holds:

$$\phi(z, x_{n+1}) + 2\lambda \langle Ax_n - Ax_{n+1}, x_{n+1} - z\rangle + \mu\lambda L\phi(x_{n+1}, x_n) \leq$$
$$\leq \phi(z, x_n) + 2\lambda \langle Ax_{n-1} - Ax_n, x_n - z\rangle + \mu\lambda L\phi(x_n, x_{n-1}) -$$
$$- (1 - 2\mu\lambda L)\phi(x_{n+1}, x_n),$$

where $z \in S$.

Theorem 1.

- C is non-empty convex and closed subset of 2-uniformly convex and uniformly smooth Banach space E;
- $A : E \to E^*$ is monotone Lipschitz continuous operator;
- $S \neq \emptyset$;
- normalized duality mapping J is sequentially weakly continuous.

Then

$$x_n \rightharpoonup z \in S.$$

ε-solution

Definition 1. $z \in C$ is ε-solution if

$$Gap(z) \leq \varepsilon,$$

where

$$Gap(z) = \sup_{y \in C} \langle Ay, z - y \rangle .$$

Theorem 2.

$$Gap(z_N) \leq \frac{\sup_{y \in C} \phi(y, x_1)}{2N},$$

where $z_N = \frac{1}{N} \sum_{i=2}^{N+1} x_i$.

Self-Adaptive Forward-Reflected-Backward

Algorithm 2

Init. Choose $x_0 \in E$, $x_1 \in E$, $\tau \in \left(0, \frac{1}{2\mu}\right)$ and λ_0, $\lambda_1 > 0$. Let $n = 1$.

1. Calculate

$$x_{n+1} = \Pi_C J^{-1}\left(Jx_n - \lambda_n Ax_n - \lambda_{n-1}\left(Ax_n - Ax_{n-1}\right)\right).$$

2. If $x_{n-1} = x_n = x_{n+1}$, then STOP, else go to 3.
3. Calculate

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \tau \frac{\|x_{n+1} - x_n\|}{\|Ax_{n+1} - Ax_n\|_*}\right\}, & \text{if } Ax_{n+1} \neq Ax_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Let $n := n + 1$ and go to 1.

FRB for equation with monotone operators

$$Ax = f, \quad f \in E^*$$

Algorithm 3

Init. Choose $x_0 \in E$, $x_1 \in E$, $\tau \in \left(0, \frac{1}{2\mu}\right)$ and λ_0, $\lambda_1 > 0$. Let $n = 1$.

1. Calculate

$$Jx_{n+1} = Jx_n - \lambda_n(Ax_n - f) - \lambda_{n-1}\left(Ax_n - Ax_{n-1}\right).$$

2. If $x_{n-1} = x_n = x_{n+1}$, then STOP, else go to 3.
3. Calculate

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \tau \frac{\|x_{n+1} - x_n\|}{\|Ax_{n+1} - Ax_n\|_*}\right\}, & \text{if } Ax_{n+1} \neq Ax_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Let $n := n + 1$ and go to 1.

Analysis 2

Lemma 2. The following inequality holds:

$$\phi(z, x_{n+1}) + 2\lambda_n \langle Ax_n - Ax_{n+1}, x_{n+1} - z\rangle + \tau\mu\frac{\lambda_n}{\lambda_{n+1}}\phi(x_{n+1}, x_n) \leq$$

$$\leq \phi(z, x_n) + 2\lambda_{n-1}\langle Ax_{n-1} - Ax_n, x_n - z\rangle + \tau\mu\frac{\lambda_{n-1}}{\lambda_n}\phi(x_n, x_{n-1}) -$$

$$- \left(1 - \tau\mu\frac{\lambda_{n-1}}{\lambda_n} - \tau\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\phi(x_{n+1}, x_n),$$

where $z \in S$.

Theorem 3.

- C is non-empty convex and closed subset of 2-uniformly convex and uniformly smooth Banach space E;
- $A : E \to E^*$ is monotone Lipschitz continuous operator;
- $S \neq \emptyset$;
- normalized duality mapping J is sequentially weakly continuous.

Then

$$x_n \rightharpoonup z \in S.$$

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Thanks for your attention!