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Design of stable periodic regimes for one class of hybrid planar systems

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The rapid development of modern science and technology requires constant attention to the study of nonlinear dynamical systems in which there are short-term processes or which are under the action of external forces, the duration of which can be neglected in the preparation of appropriate mathematical models.

Such systems are found, for example, in mechanics, chemical technology, medicine and biology, aircraft dynamics, mathematical economics, control theory and other fields of science and technology, where we have to study systems under the influence of short-term (pulsed) external forces called systems with pulsed action.

In fact, it has been found that the presence of impulse action can significantly complicate the behavior of the trajectories of such systems, even for cases of relatively simple differential equations. In the general case, in the presence of impulse action, the behavior of solutions of differential equations (even linear differential equations with constant coefficients) can be significantly nonlinear and significantly different from the behavior of such systems in the absence of impulse action.

Studying the nonlinear damping of oscillations in electric circuits, Lenard obtained a natural generalization of the famous van der Paul equation. At the same time, the problem of the existence of periodic regimes is important for oscillating systems in the region. Note that the boundary cycle is an isolated closed trajectory of the vector field (in other words, it is a periodic solution in some neighborhood which has no other periodic solutions, respectively, all other trajectories from this region are wound on the boundary cycle in positive or negative time). . Therefore, when modeling many systems of oscillating systems that are affected by destabilizing factors of instantaneous (impulse) nature, it is important to understand the conditions for the existence of stable periodic regimes in them. Obtaining appropriate design conditions allows to develop methods to support decision-making on the management of such systems.

Let us investigate the problem of the existence of harmonic cycles for the Lénard equation, the solutions of which are subjected to instantaneous forces of impulse nature and at unfixed moments in time.

Consider a dynamical system in which motion is described by a generalized Lénard differential equation of the form:

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0, \qquad (1)$$

($x \in \mathcal{D} \subset \mathbb{R}^3$, $\mathcal{D} -$ phase space of the system (1), $t \in \mathbb{R} -$ time) and which is affected by instantaneous forces determined by some operator $\mathcal{A}_t$, that at the moment of passing a moving point of some fixed position$x = x_*$ acts according to the rule $(x,\ t) \to (t,\ \mathcal{A}_t x)$. Impulse action in such a system occurs at non-fixed moments of time and increases the amount of motion in the system by a certain amount $I(\dot{x})$, which depends on the speed of the moving point at the time of its passage $x = x_*$. Next we will consider that $I(y)$, де $y = \dot{x}$, as a function of its argument is continuous.

If t_* some point in time at which the moving point reaches position $x = x_*$, when a pulsed action occurs, the pulsed perturbations of the moving point are recorded

$$\Delta \frac{dx}{dt}\bigg|_{x=x_*} = \frac{dx}{dt}\bigg|_{t=t_*+0} - \frac{dx}{dt}\bigg|_{t=t_*-0} = \mathcal{A}_t x - x = I(\dot{x}). \qquad (2)$$

Equation (1) is written in an equivalent way as a system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x, y)y. \end{cases} \qquad (3)$$

In the future we will use the notation

$$G(x) = \int_0^x g(s)\,ds.$$

We will consider further that functions $g(x)$ and $f(x,y)$ provide the condition for the existence and uniqueness of the solution of the system (3). In addition, we will assume that

$$x\,g(x) > 0 \quad \text{при} \quad x \neq 0 \tag{4}$$

and

$$G(\pm\infty) = \int_0^{\pm\infty} g(x)\,dx. \tag{5}$$

$$w(x,y) \equiv G(x) + \frac{1}{2}y^2 . \tag{6}$$

They are all closed. At pair function $g(x)$,

$$g(-x) = -g(x) \tag{7}$$

Let's mark $y_* = \sqrt{2G(x_*)}$.

Under such conditions, studying the motion of the phase point of system (1), (2), we construct a Poincare map for the line $x = x_*$, which is used to study the question of the existence of periodic solutions of problem (1), (2). It is obvious that in this case the problem of the existence of periodic solutions of system (1), (2) is reduced to the problem of the existence of periodic and fixed points of some mapping of a segment into the same segment.

Let us investigate further the case when in some domain there are boundary cycles, ie there are isolated periodic solutions of the system, so in this domain $f(x,y) \not\equiv 0$. The problem of the existence and uniqueness of boundary cycles for system (2) is considered in the following theorems.

Theorem 1. *Let $f(0,0) < 0$ and*

$$f(x,y) \geq 0 \quad \textit{for} \quad |x| \geq x_0 > 0,$$

moreover

$$f(x,y) \geq -\mathcal{F} \ \textit{for} \ |x| \leq x_0. \tag{8}$$

Let there be a meaning $x_1 > x_0$, such that

$$\int_{x_0}^{x_1} f(x,y)\,dx \geq 10\mathcal{F}x_0, \tag{9}$$

where $y = y(x) > 0$ — arbitrary continuously decreasing function. (Obviously the value x_1 can be arbitrarily increased). Under such conditions, system (3) has at least one periodic solution.

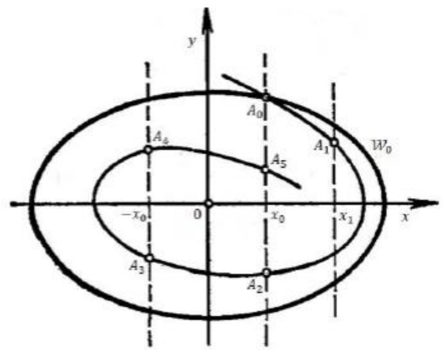

Figure 1: Illustration of the limit cycle of the theorem

Now let us consider the uniqueness result. We denote by $\mathcal{R}_+$ and $\mathcal{R}_-$ domains in the phase plane xy, in which the function $f(x,y)$ is positive or negative. The part of the curve $w(x,y) = w$, that belongs to $\mathcal{R}_\pm$ we will denote by $\mathcal{R}_\pm(w)$.

Теорема 2. *Assume conditions of Theorem 2, and assume that the function $f(x,y)$ has continuous derivatives of the first order. Additionally, we assume that for every value of parameter w, for which the set $\mathcal{R}_\pm(w)$ exists, the infimum of the function*

$$F(x,y) = \frac{1}{y^2} + \frac{1}{yf(x,y)}\frac{\partial}{\partial y} f(x,y) \tag{10}$$

on $\mathcal{R}_+(w)$ is positive and no less than its supremum on $\mathcal{R}_-(w)$. Then system (3) has a unique limit cycle.

Let us consider the behavior of system (3) under assumptions of theorem **1,2**.

$$w_0 = G(x_0) + \frac{1}{2} y_0^2.$$

Assume that

1^0. Generalized Liénard differential equation (1) satisfies the conditions of existence and uniqueness of solution.

2^0. The line $x = x_*$ is transversal to the flow (1) everywhere everywhere except the trajectory for which the line $x = x_*$ is tangent. And in this case we assume that $I(0) = 0$.

3^0. The impulsive operator $\mathcal{A}_t$ is continuous with respect to veriables $(x, \dot{x})$.

Let for the initial data $(x_0, \dot{x}_0)$ of the problem (3), the following property takes place

$\mathbb{M}$: $x_0 < x_*$ and $\dot{x}_0 > x_*$, or $x_0 > x_*$ and $\dot{x}_0 < x_*$, or $(x_0, \dot{x}_0) \in w_0$ i $|x_0| > x_*$

and conditions of theorems **1,2** hold for all (x, y). Then the phase point $(x_0, \dot{x}_0)$ will move along trajectory $x(t_*, x_0, \dot{x}_0, t_0) = x_*$, when it is affected by impulsive perturbation (12). Let

$$t_1 = \min_{t_* > t_0} \{t_* : x(t_*, x_0, \dot{x}_0, t_0) = x_*\}.$$

Let us consider coordinates of the phase point $\big(x(t), \dot{x}(t)\big)$, where $x(t) = x(t, x_0, \dot{x}_0, t_0)$ for $t = t_1 + 0$, i.e., afterimpulsive perturbation. Then

$$\begin{aligned} x(t_1 + 0) &= x_*, \\ \dot{x}(t_1 + 0) &= \dot{x}(t_1, x_0, \dot{x}_0, t_0) + I\big(\dot{x}(t_1, x_0, \dot{x}_0, t_0)\big). \end{aligned} \qquad \textbf{(11)}$$

Then, if $(x_1, \dot{x}_1)$, when $x_1 = x_*$, and $\dot{x}_1 = \dot{x}(t_1 + 0)$ is defined by (**11**), as a new initial data for the problem (3), the property $\mathbb{M}$ holds, i.e., $(x_*, \dot{x}_1) \in w_*$, then there exists a moment of time t_* such that $x(t_*, x_*, \dot{x}_0, t_0) = x_*$, when the phase point of (3), is again affected by impulsive perturbations. We denote

$$t_2 = \min_{t_* > t_1} \{t_* : x(t_*, x_*, \dot{x}_1, t_1) = x_*\}.$$

Note, that if the property $\mathbb{M}$ does not hold, i.e., $(x_*, \dot{x}_1) \notin w_*$, then the phase point of the system (3), for $t > t_1$ will not have impulsive perturbations. This situation is possible only if $x_* = x'_*$, when the hyperplane does not intersect the limit cycle W_0 at any point. In such a case the system (3), has only one impulsive influences. After that we get a new initial data for (3) which has a limit cycle due to theorem **1**,**2**, and the phase point in the sequel will move without any impulses.

Assume that we have constructed n members of the sequence $\{t_k, (x_k, \dot{x}_k)\}$, $k = \overline{1, n}$, where

$$t_1 = \min_{t_* > t_{k-1}} \{t_* : x(t_*, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) = x_*\}, \tag{12}$$

$$x_k = x(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) = x_*, \quad k = \overline{1, n}, \tag{13}$$

$$\dot{x}_k = \dot{x}(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1}) + I\big(\dot{x}(t_k, x_{k-1}, \dot{x}_{k-1}, t_{k-1})\big). \tag{14}$$

It is clear that $(x_k, \dot{x}_k) = (x_*, \dot{x}_k) \in w_*$, where $k = \overline{1, n-1}$. Under condition $(x_n, \dot{x}_n) = (x_*, \dot{x}_n) \in w_*$, it is possible to construct $(n+1)$-th member of this sequence. Otherwise, it consists of only n members.

In general case, for arbitrary values of initial data $(x_0, \dot{x}_0)$ the sequence $t_1, t_2, \dots$ can be infinite, or it can be finite, in particular, it may consists of only one element, or it can be empty.

If the sequence of moments of time consists of one point, which is possible, for example, for the case when for all $x \notin w_*$, then the system (**3**), undergoes an impulse action only once and for it there is a limit cycle in the requirements of Theorems **1**,**2** .

If the sequence has a finite (not empty) number of points containing, for example, exactly k≥1 elements, then the condition $|\dot{x_k} + I(\dot{x_k})| \notin w_*$, $I(w_*) = 0$, and the system has exactly k times the action of impulses, and there is a limit cycle in the requirements of Theorems 1,2.

It easy to see that if for some n and for all $y \in [-y_*, y_*] \cup (-\infty, -y_*) \cup (y_*, \infty)$ we have $|f^n(y)| = w_*$, where $f^n(y)$ is the n-th iteration of the function $f(y) = -y - I(-y)$, $y = \dot{x}$, then the sequence $\{t_n, (x_n, \dot{x}_n)\}$, $n \in \mathbb{N}$ has an infinite number of points. Moreover, $(x_n, \dot{x}_n) = (x_*, \dot{x}_n) = w_*$ for all k.

It follows from the analysis that problem (3), will have a single limit cycle under the conditions of Theorem 2 where the phase point will be affected by impulse perturbation when the sequence $\{t_n\}$, $n = 1,2, \ldots$ is infinite. The following theorem takes place

Theorem 4. *Assume that*

1) Function $f(x, y)$ has continuous derivatives of the first order, and, moreover, there exist positive x_1, x_2 such that

$f(x, y) < 0, \quad \inf f(x, y) = -\mathcal{F}$ *on* (x_1, x_2)

and

$f(x, y) \geq 0$ *otherwise;*

2) $y\dfrac{\partial y}{\partial x} \geq 0$

and $\int_{x_0}^{x_2} f(x, y)dx \geq 10\mathcal{F}x_0$,

where $x_0 = \min(x_1, x_2)$ *is sufficiently large and* $y = y(x)$ *is an arbitrary nonincreasing function;*

3) $G(-x_1) = G(x_2)$.

Then the impulsive system (3) has a unique limit cycle, and the phase point has $n \in \mathbb{N}$ impulsive perturbations.

The analysis of the qualitative behavior of the system (1), (2), demonstrates the complex behavior of the generalized Lénard equation (1) with impulse action (2). The conditions necessary and sufficient conditions for the existence of a single stable limit cycle for such an equation are investigated.

The paper investigates the conditions for the existence of boundary cycles for the Lienard equation, the solutions of which are affected by instantaneous forces of momentum nature at unfixed moments in time.

For this system, the constructive conditions for the existence of $T(n)$-periodic solutions are obtained such that the phase point of the system when moving along the corresponding trajectory undergoes exactly n pulse actions for the period where n is an arbitrary natural number. The points that define cycles that correspond to periodic solutions satisfy the Sharkovsky order.

The conditions for the existence of at least one periodic solution and a single limit cycle are found — the only one with precision to shift in time of the periodic solution.

For system (3) with pulsed action the existence of a single stable limit cycle is proved, the phase point of which will be affected by pulsed forces $n \in \mathbb{N}$ times. It is shown that a stable limit cycle under given conditions will exist despite the influence of impulse forces according to the law.

Thanks for your attention

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