An Approach for Improving Local Solutions in the Unequal Circles Packing Problem

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Introduction

The problems of packing circles and spheres differ in the space dimension, the shape of containers, the specifics of accounting for metrical parameters (sizes) of objects and containers, etc. The number of publications in this area has increased.

On the one hand, is due to the fact that the problems have numerous practical applications. On the other hand, the ones have become a testing benchmark for new methods of global optimization under test, since they are characterized as multi-extremum and of high dimension.

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According to the typology proposed by Wäscher G., Haußner, H., Schumann H. ("An improved typology of cutting and packing problems," European Journal of Operational Research 183, 1109– 1130 (2007) the problems can be considered as the Knapsack Problems (KP) or Open Dimension Problems (ODP): 2DCKP, 3DSKP, 2DCODP, 3DSODP.

The subject of this talk is the problem of packing unequal circles into a circle of minimum radius. However, the approach can be easily extended to packing both circles and spheres into an arbitrary container and even into multiple containers.

PROBLEM STATEMENT

Let $S_1, S_2, \ldots, S_n$ be a set of n circles, the radii of which are equal $\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n$ respectively.

Without loss of generality, suppose

$$\hat{r}_1 \le \hat{r}_2 \le \ldots \le \hat{r}_n.$$

The problem is to place the circles $S_1, S_2, \ldots, S_n$ into a circle S_0 of minimum radius r_0.

Denote the coordinates of the circle centers by $p^i = (u_i, v_i)$, $i \in J_n = \{1, 2, \ldots, n\}$ and put $p^0 = (0, 0)$. Then the mathematical model of the problem has the form

$$r_0 \to min \qquad (1)$$

subject to

$$u_i^2 + v_i^2 \le (r_0 - \hat{r}_i)^2, \qquad (2)$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + \hat{r}_j)^2, \qquad (3)$$

$$i \in J_n,\ j \in J_n,\ i < j.$$

We have a nonlinear optimization problem with variables $r_0, u_i, v_i, i \in J_n$, which call **Problem 1.**

Problem 1 is multiextremal due to the nonconvexity of constraints (3). The use of numerical methods for nonlinear optimization allows finding only its local solutions, depending on the choice of initial values $r_0,\ u_i, v_i,\ i \in J_n$.

Let us consider the radii as independent variables and form the system

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} \hat{r}_i \ , \tag{4}$$

$$\sum_{i \in W} r_i \geq \sum_{i=1}^{|W|} \hat{r}_i \quad \forall W \subset J_n, \tag{5}$$

$$\sum_{i=1}^{n} \left(r_i - \tau \right)^2 = \sum_{i=1}^{n} \left(\hat{r}_i - \tau \right)^2, \tag{6}$$

$$\tau = \frac{1}{n} \sum_{i=1}^{n} \hat{r}_i . \tag{7}$$

System (4) - (6) is such that the set of its solutions coincides with the permutation set of real numbers $\{\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n\}$.

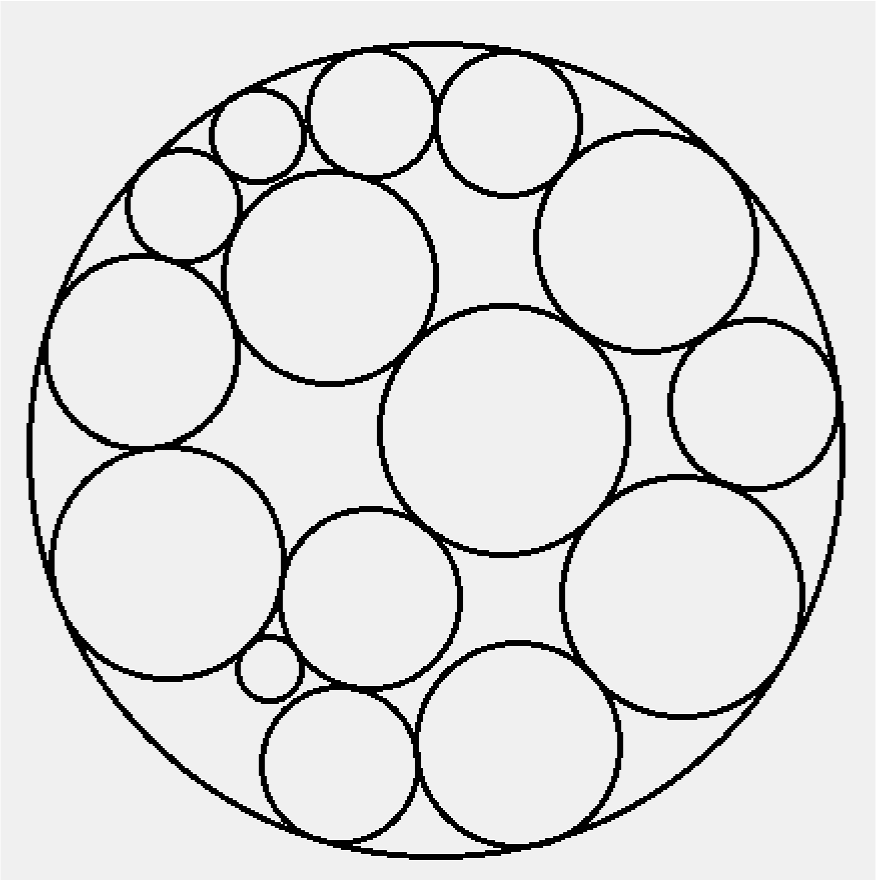
Representation (4) - (6) is called polyhedral-spherical, because it describes a set that is the intersection of the permutation polyhedron (4) - (5) and the hypersphere (6) with center at point (7).

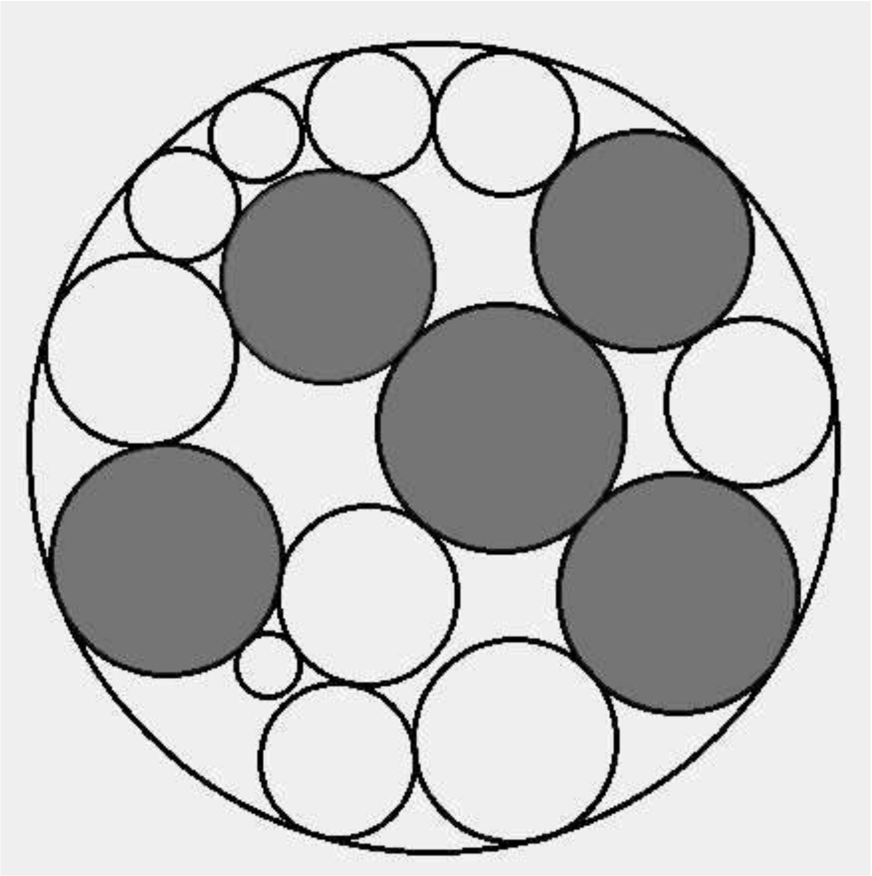
Theorem 1. *Both Problem 1 and Problem 2 are equivalent in the sense of the coincidence of their global solutions.*

Problem 1 has a dimension $2n+1$, and the dimension of *Problem 2* is equal to $3n+1$.

We have implemented an approach of artificial expansion of the space of variables. Applied to the circle and sphere packing problem, this approach will be called the variable radius method.

The main difficulties in implementing the proposed approach are associated with an increasing of dimension of *Problem 2*. The number of linear constraints in system (5) is equal to $2^n - 2$, which already at $n > 15$ leads to difficulties in applying numerical optimization methods. Using the properties of linear function on combinatorial polyhedra allows one to partially overcome the arising difficulties.





The method of variable radii allows one to realize the idea of improving solutions of *Problem 1*. Indeed, we choose the local solution obtained as the starting point and consider the radii of circles as independent variables $r_1, r_2, \ldots, r_n$ satisfying constraint system (4) - (9).

As a result, it is possible by variables to overcome the neighborhood of attraction of a local extremum and direct to a new, may be better, local solution.

Consider the problem of minimizing the function (1) subject to

$$u_i^2 + v_i^2 \le (r_0 - r_i)^2, \tag{8}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (r_i + r_j)^2, \tag{9}$$

$$i \in J_n, j \in J_n, i < j,$$

where variables r_i, satisfy conditions (4) - (7).

The nonlinear optimization problem (1), (4) - (9) we call **Problem 2.**

Put $l < n$ and select l circles with variables radii. Let $M' = \{m_1, m_{2,\ldots,} m_l\} \subset J_n$ be ordered numbers of circles, whose radii $\hat{r}_{m_1} \le \hat{r}_{m_2} \le \ldots \le \hat{r}_{m_l}$ are fixed, and $M'' = J_n \setminus M'$ be of ordered numbers of circles with variables radii: r_i, $i \in M''$.

Form a system

$$u_i^2 + v_i^2 \le (r_0 - \hat{r}_i)^2, \tag{10}$$

$$u_i^2 + v_i^2 \le (r_0 - r_i)^2,\ i \in M'' \tag{11}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + \hat{r}_j)^2 \tag{12}$$

$$i \in M',\ j \in M',\ i < j$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (r_i + r_j)^2 \tag{13}$$

$$i \in M'',\ j \in M'',\ i < j,$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + r_j)^2, \tag{14}$$

$$i \in M',\ j \in M''.$$

$$\sum_{i \in M''} r_i = \sum_{i \in M''} \hat{r}_i \tag{15}$$

$$\sum_{i \in W} r_i \geq \sum_{i=1}^{|W|} \hat{r}_{m_i} \quad \forall W \subset M'' \tag{16}$$

$$\sum_{i \in M''} \left(r_i - \hat{\tau} \right)^2 = \sum_{i \in M''} \left(\hat{r}_i - \hat{\tau} \right)^2 \tag{17}$$

$$\hat{\tau} = \frac{1}{l} \sum_{i=1}^{l} \hat{r}_{m_i} . \tag{18}$$

The problem (1), (10) - (18) of variables r_0, u_i, v_i, $i \in J_n$, and r_j, $j \in M''$ we call **Problem 3.**

Theorem 2. *The sets of global solutions of both Problem1 and Problem 3 are the same for any* $M' = \{m_1, m_2, \ldots, m_l\} \subset J_n$.

Consider a set $Q = \{r_1, r_2, \ldots, r_n\}$ and part it onto $q+1$ pairwise disjoint subsets as follows. Let be

$$\bigcup_{k=0}^{q} M^k = J_n, \quad M^k = \{m_1, m_2, \ldots, m_{l_k}\} \tag{19}$$

$$Q = \bigcup_{k=0}^{q} Q^k \tag{20}$$

$$Q^0 = \{\hat{r}_{m_1}, \ldots, \hat{r}_{m_{l_0}}\}, Q^k = \{r_{m_1}, \ldots, r_{m_{l_k}}\}, \ k \in J_q$$

$$u_i^2 + v_i^2 \le (r_0 - \hat{r}_i)^2,\ i \in M^0 \tag{21}$$

$$u_i^2 + v_i^2 \le (r_0 - r_i)^2,\ i \in J_n \setminus M^0 \tag{22}$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + \hat{r}_j)^2, \tag{23}$$

$$i \in M^0,\ j \in M^0,\ i < j,$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (r_i + r_j)^2, \tag{24}$$

$$i \in J_n \setminus M^0,\ j \in J_n \setminus M^0,\ i < j,$$

$$(u_i - u_j)^2 + (v_i - v_j)^2 \ge (\hat{r}_i + r_j)^2, \tag{25}$$

$$i \in M^0,\ j \in J_n \setminus M^0.$$

For each set Q^k, $k \in J_q$, we write

$$\sum_{i \in M^k} r_i = \sum_{i \in M^k} \hat{r}_i \tag{26}$$

$$\sum_{i \in W} r_i \geq \sum_{i=1}^{|W|} \hat{r}_{m_i}, \quad \forall W \subset M^k \tag{27}$$

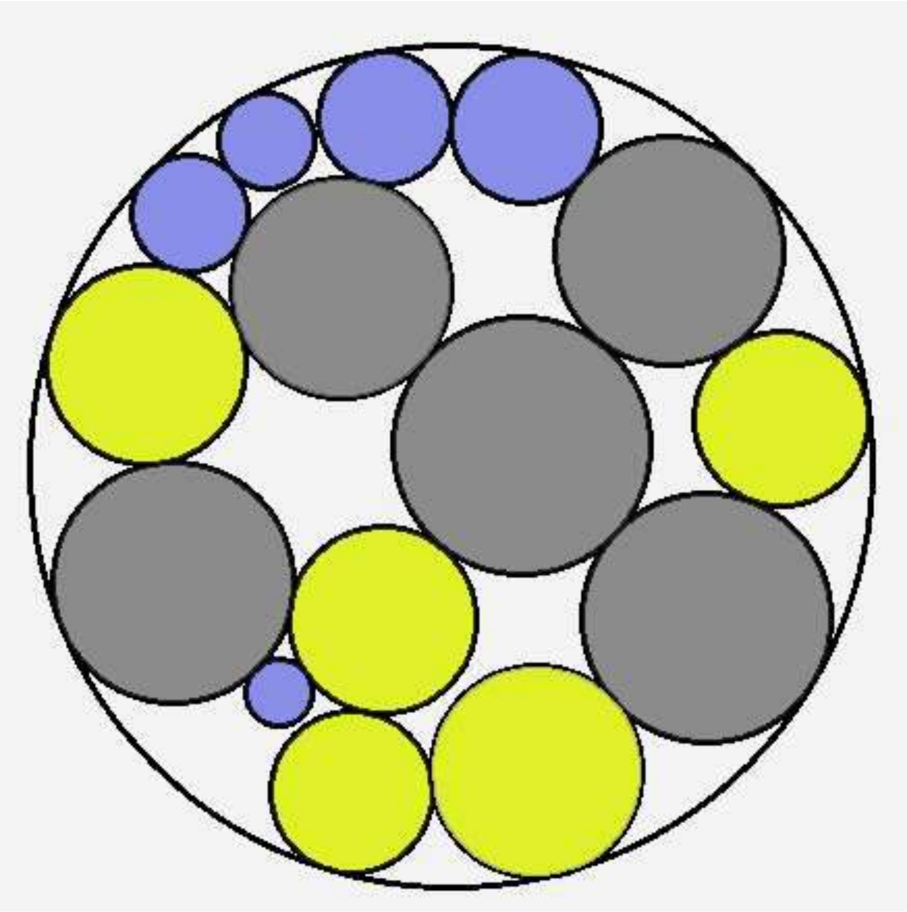
$$\sum_{i \in M^k} \left(r_i - \tau_k\right)^2 = \sum_{i \in M^k} \left(\hat{r}_i - \tau_k\right)^2, \tag{28}$$

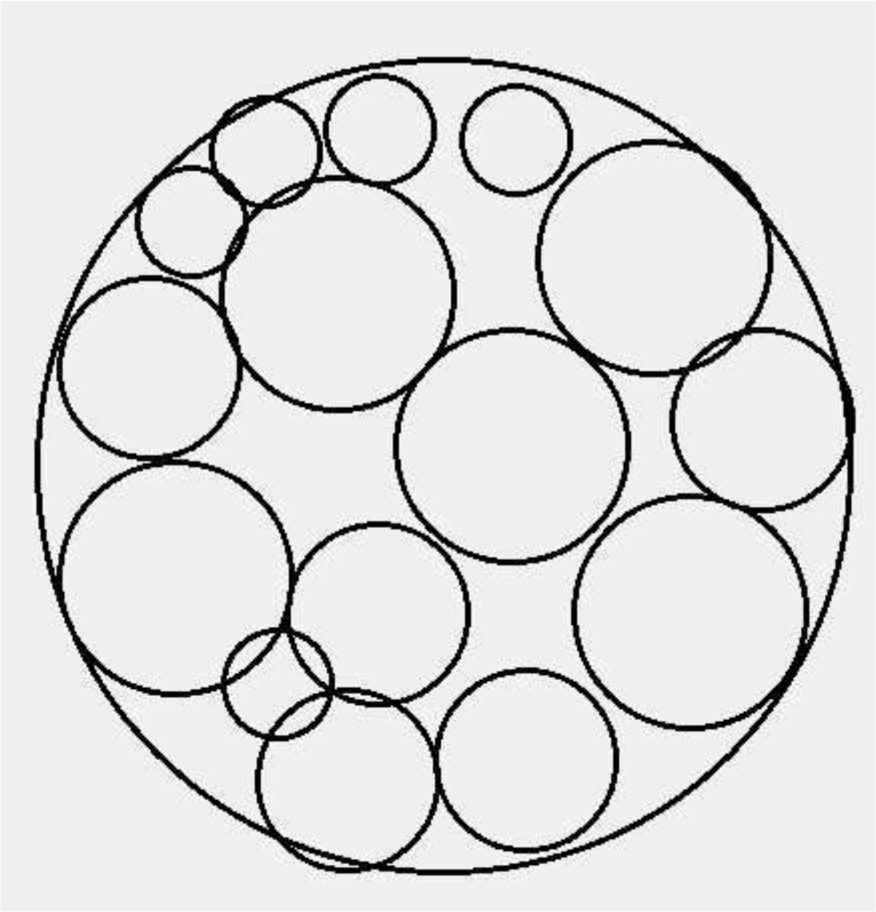
$$\tau_k = \frac{1}{l_k} \sum_{i=1}^{l_k} \hat{r}_{m_i}. \tag{29}$$

The problem (1), (21) - (29) in the space of variables $r_0, u_i, v_i, i \in J_n$, and $r_j,\ j \in J_n \setminus M^0$ we call *Problem 4.*

Theorem 3. *The sets of global solutions of both Problem 1 and Problem 4 are the same for any* $M^0 = \left\{m_1, m_2, \ldots, m_{l_0}\right\} \subset J_n$.

The choice of the method of partitioning $Q = \left\{r_1, \ldots, r_n\right\}$ onto a sets $Q^k = \left\{r_{m_1}, \ldots, r_{m_{l_k}}\right\}$, $k \in J_q^0$ with the subsequent formation of constraints (12) - (14), forms a family of modifications of the method of variable radius.

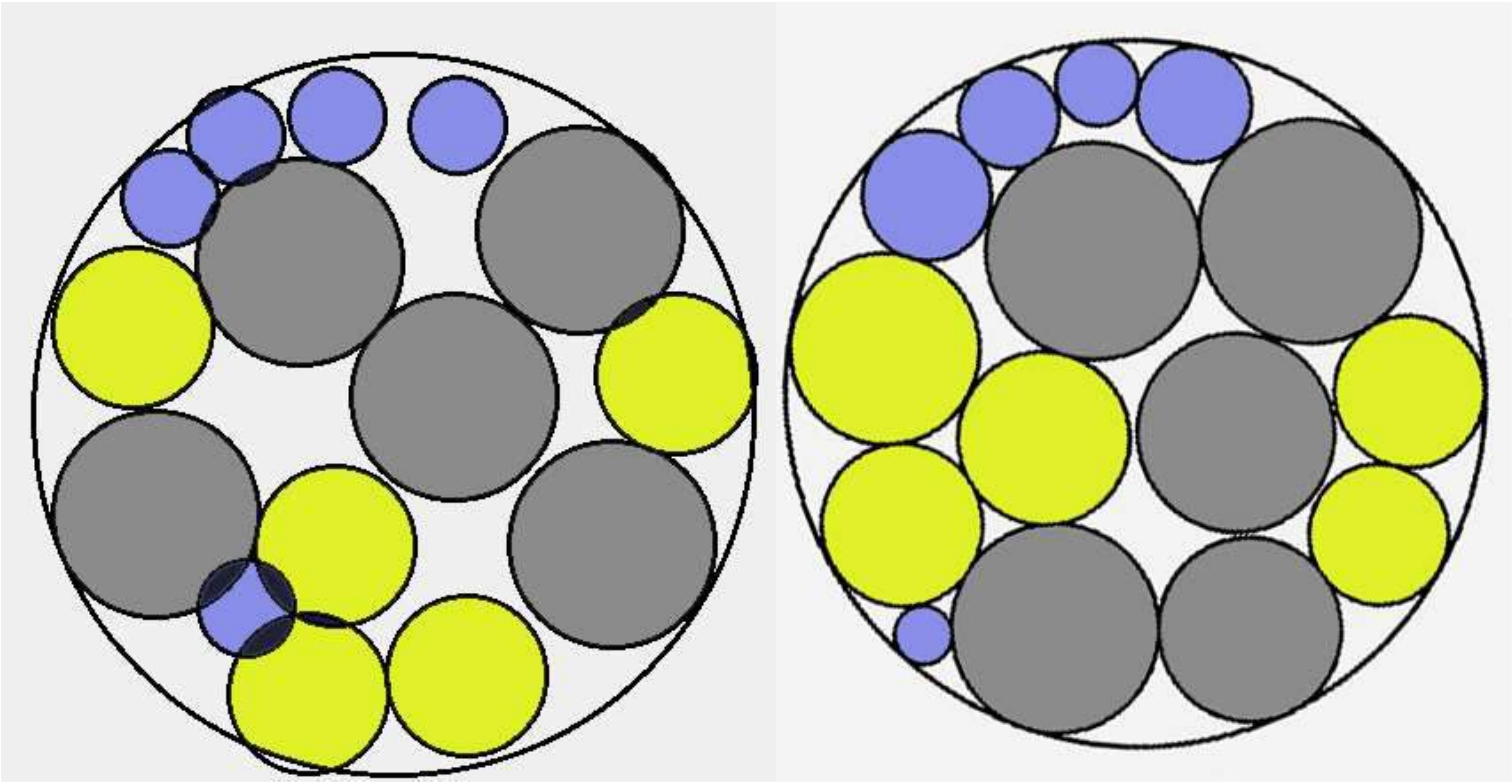




Numerical Results

We conducted the following computational experiments. For local optimization, the IPOPT software package was used *(https://projects.coin-or.org/Ipopt)*, which implements an internal point method for continuous nonlinear programming problems. To perform the calculations was used PC with the characteristics i3/8G/SSD 256G.

At first, test problems with the number of circles less than 15 were considered. A series of 30 problems was formed in which the radii of the circles were randomly generated uniformly in the interval (1, 15).



For each test, the coordinates of the circle centers from (-100,100) x (-100,100) were generated randomly. The values obtained were chosen as initial for the implementation of the local optimization method in *Problem 1*. Then, with the same initial data, *Problem 2* was solved, in which the radii of the circles are variables. The initial values for the variable radii $r_i, i \in J_n$ in *Problem 2* were randomly generated from the interval (1, 15). Thus, we compared the solutions of both *Problem 1* and *Problem 2* using the same local optimization method for the same starting point.

For each test, the coordinates of the circle centers from (-100,100) x (-100,100) were generated randomly. The values obtained were chosen as initial for the implementation of the local optimization method in *Problem 1*. Then, with the same initial data, *Problem 2* was solved, in which the radii of the circles are variables. The initial values for the variable radii $r_i, i \in J_n$ in *Problem 2* were randomly generated from the interval (1, 15). We compared the solutions of both *Problem 1* and *Problem 2* using the same local optimization method for the same starting point.

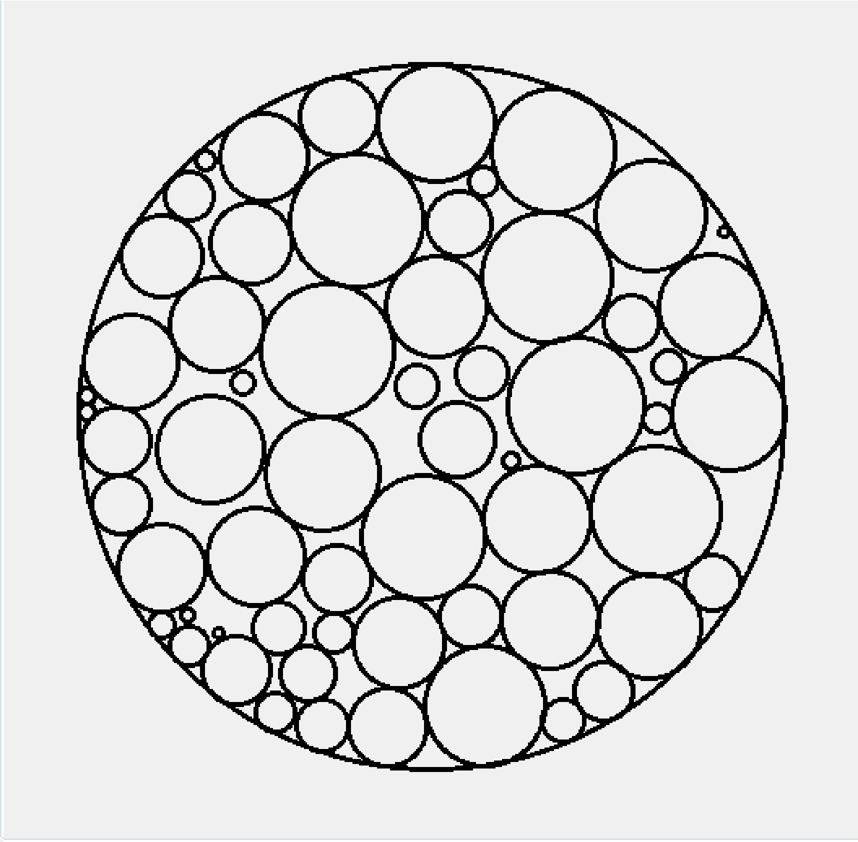
The proposed approach was tested on the placing of 60 circles, the radii of which were chosen randomly and ordered. In this way $Q=\{\hat{r}_1,\hat{r}_2,\ldots,\hat{r}_{60}\}=\{6,6,8,9,10,10,13,13,15,17,18,20,$ 23, 25, 25, 26, 26, 30, 31, 32, 32, 33, 34, 34, 35, 37, 38, 39, 41, 42, 42, 45, 48, 49, 49, 51, 55, 55, 56, 58, 58, 59, 61, 62, 63, 65, 66, 66, 69, 69, 70, 73, 73, 75, 76, 79, 80, 82, 82, 83$\}$.

First, a local solution of *Problem 1* was obtained with fixed radii from Q. The starting point was randomly generated uniformly from the square $(-500,500)\times(-500,500)$. Radius $r_0^{(0)}=450.9$ was obtained in 362 sec.

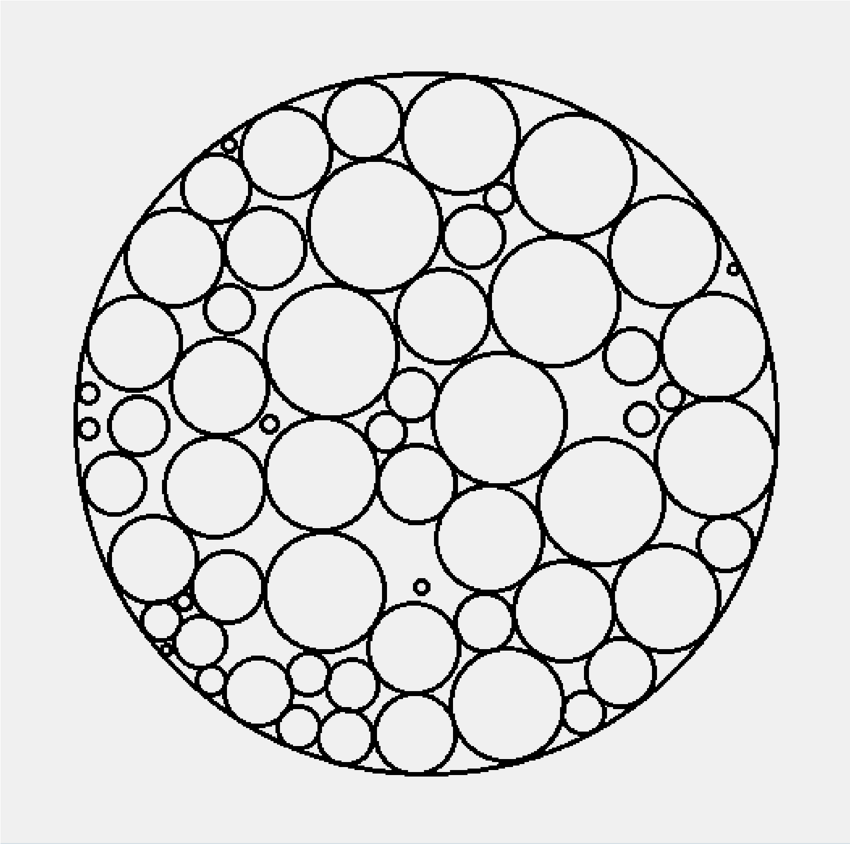
The set Q was partitioned in accordance with *Strategy 3* and *Problem 4* was solved. At each subsequent iteration, the coordinates of the centers of the circles corresponding to the best local solution were chosen as the starting point for the numerical local optimization method.

The set $Q = \{\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_n\}$ was partitioned into 10, 12, 15, 20, 30 groups of 6, 5, 4, 3, 2 elements, respectively. At subsequent iterations were obtained $r_0^{(1)} = 434.7$, $r_0^{(2)} = 433.1$, $r_0^{(3)} = 431.2$, $r_0^{(4)} = 430.9$, $r_0^{(5)} = 430.9$. Finally, $r_0^{(6)} = 426.7$. The runtime was about 17 min.

Numerical Results

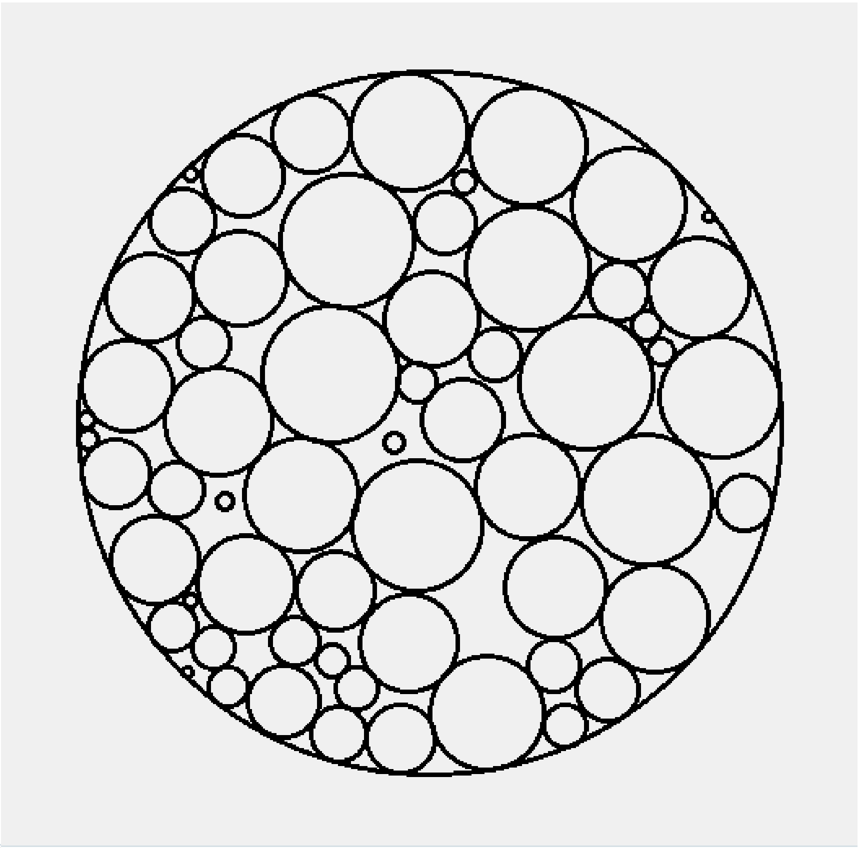


Test Variable Radius_5 Groups_by 12
R=434,72 Run time 2.31 min

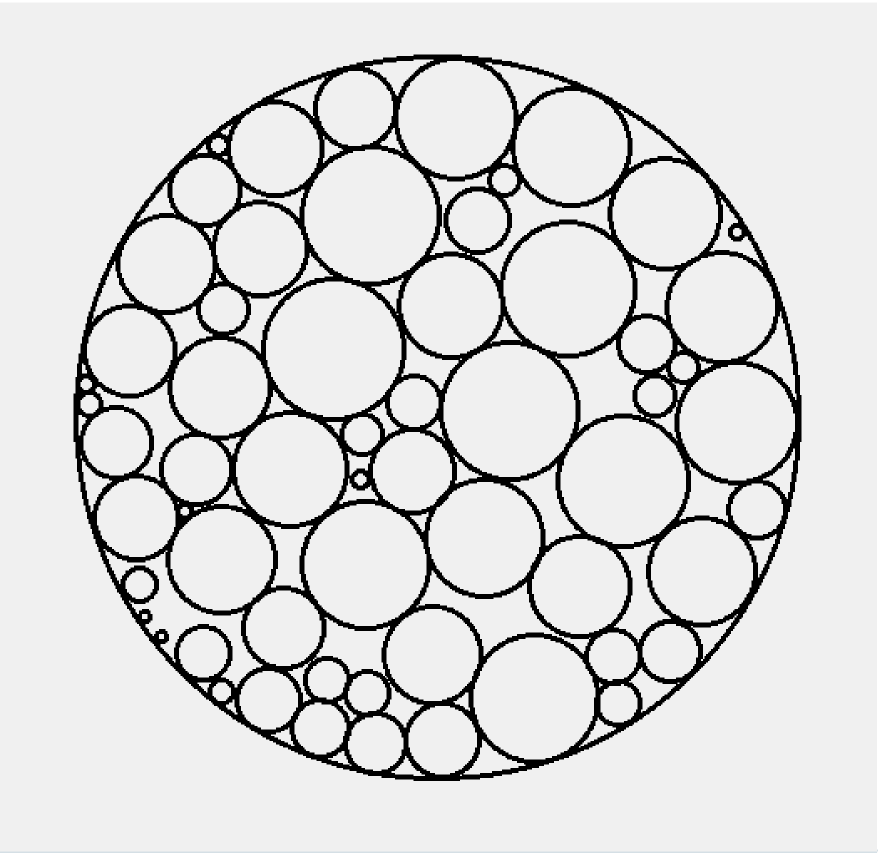


Test Variable Radius_6 Groups_by 10
R=433,15 Run time 3.44 min

Numerical Results

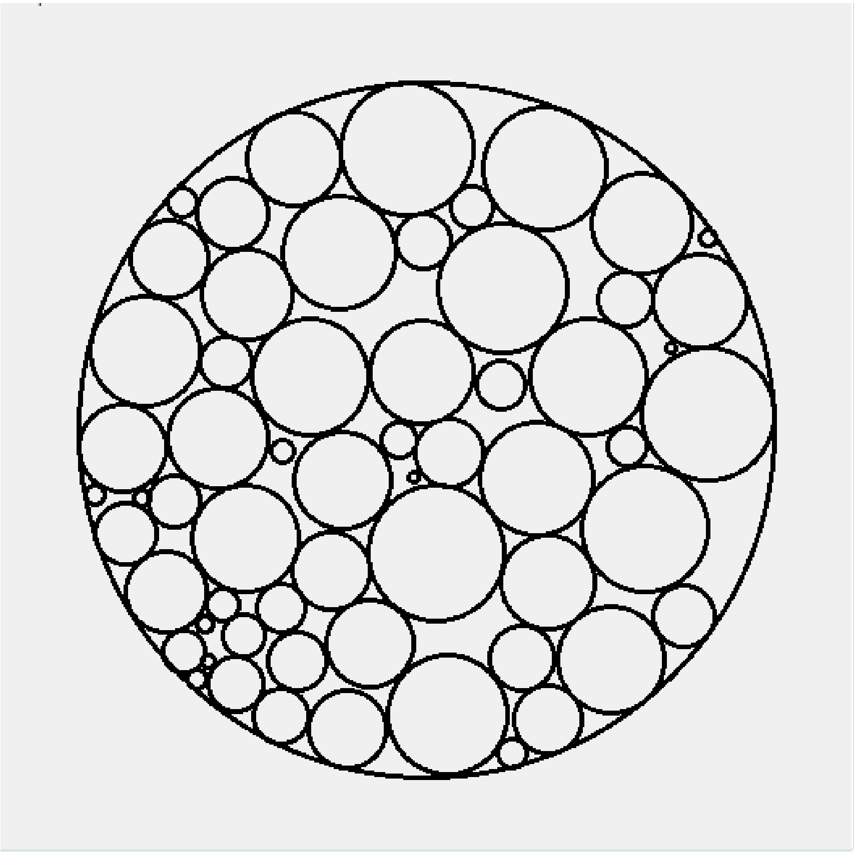


Test Variable Radius_10 Group by 6
R=432,44 Run time 7.47 min

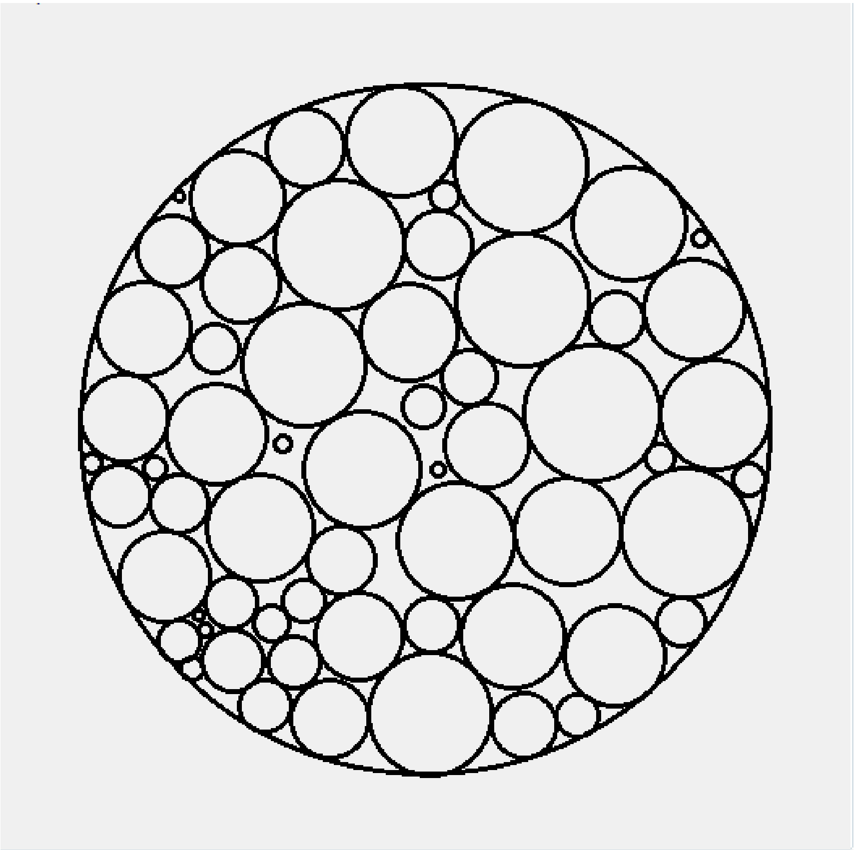


Test VariableRadius_12 Groups by 5
R=431,22 Run time 4.13 min

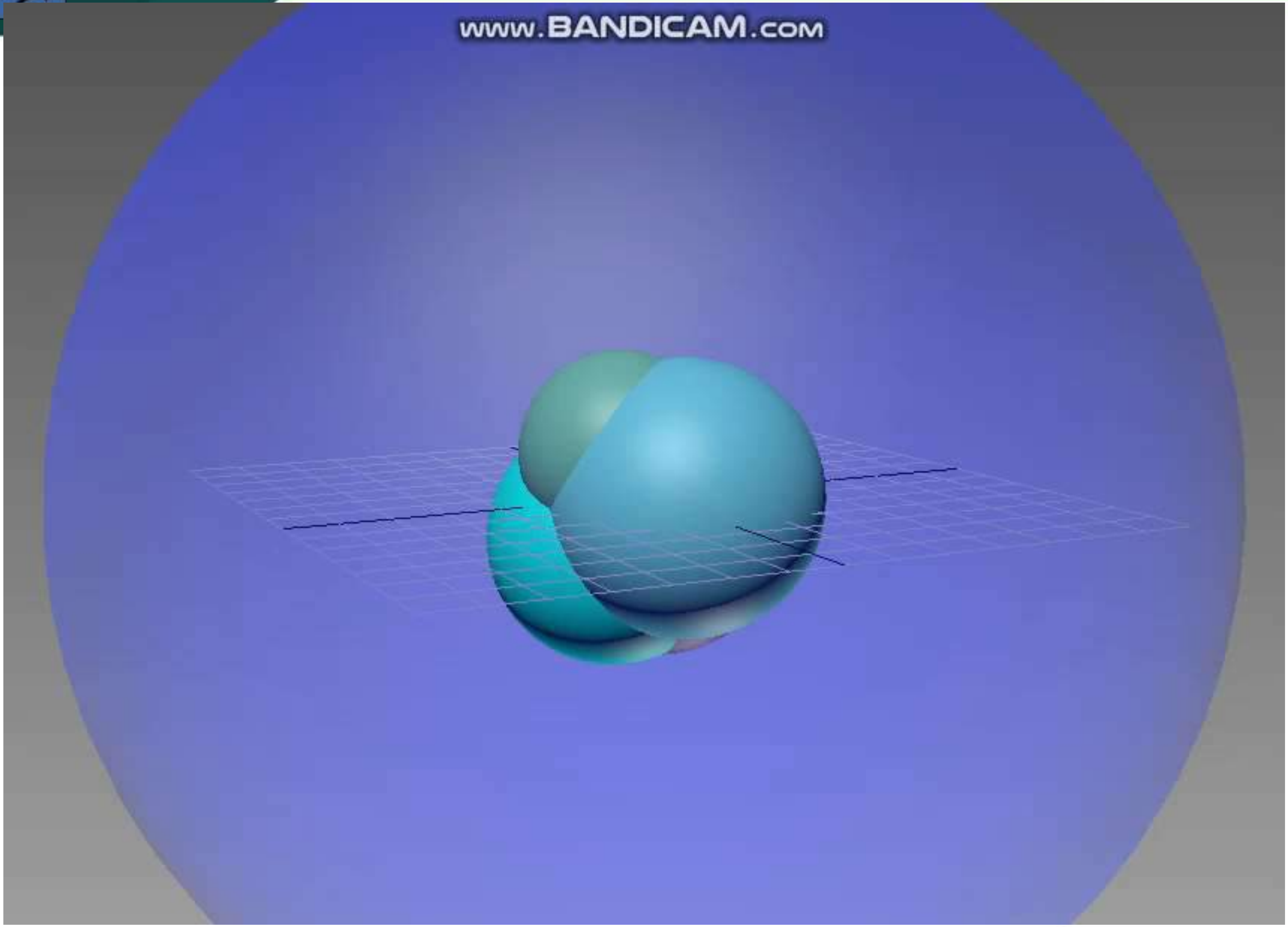
Numerical Results



Test VariableRadius_15 Groups_by 4
R=430,89 Run time 7.29 min



Test Variable Radius_20 Groups by 3
R=426,73 Run time 1.28 min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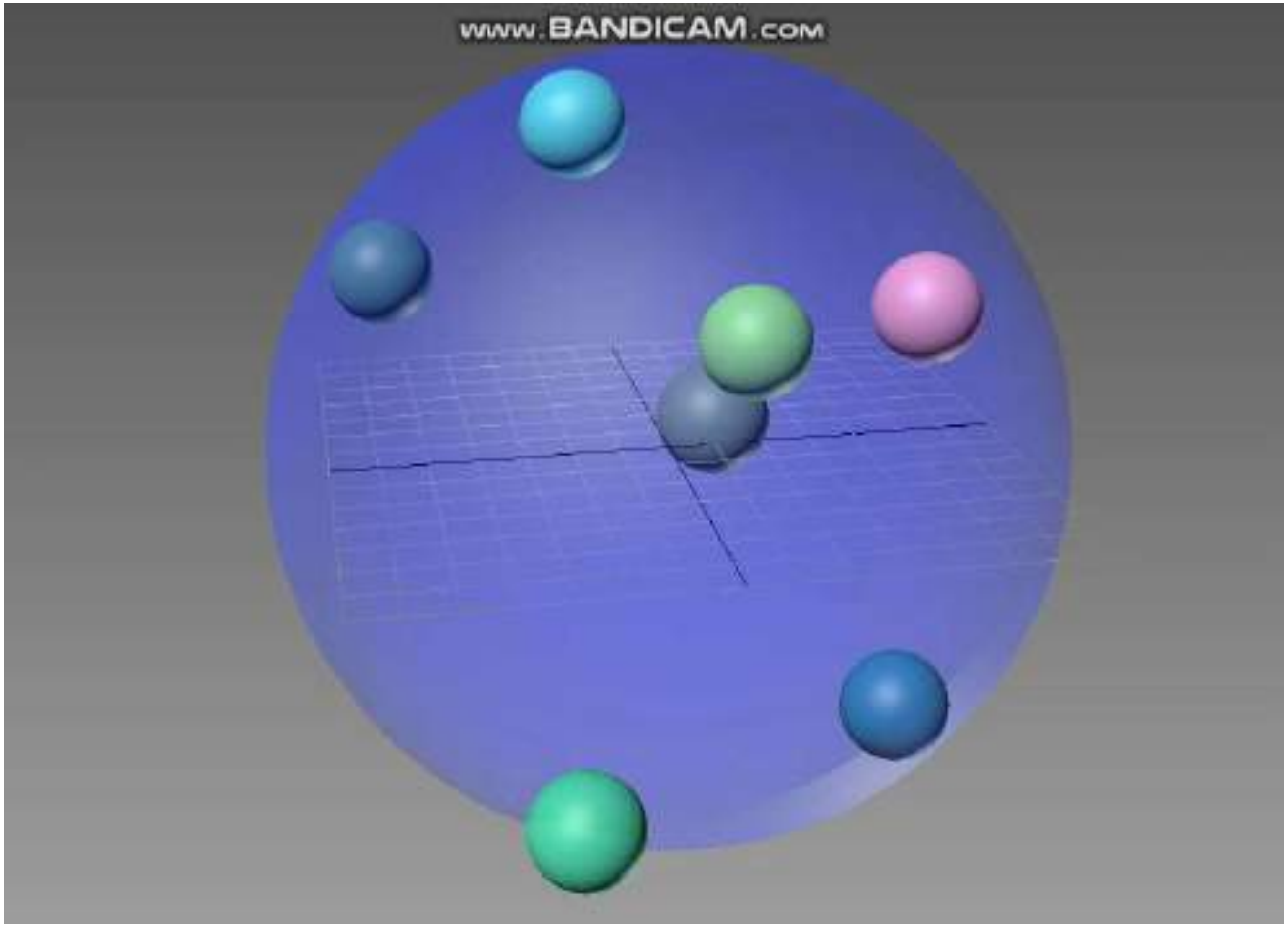
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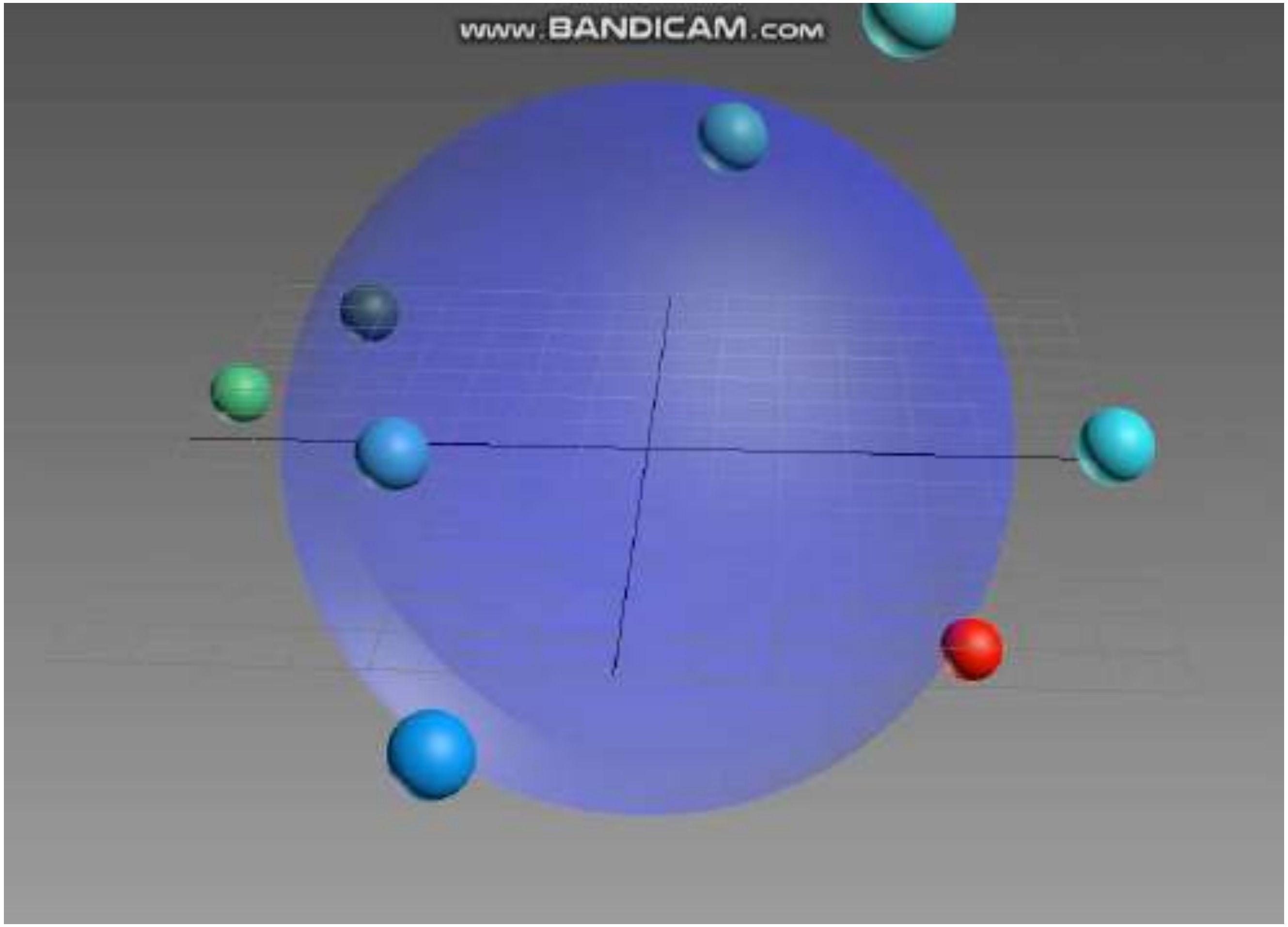
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CONCLUSIONS

A new approach was proposed to improve the local solutions of the unequal circles packing problem into a circular container of minimum radius. The basic idea is to expand the space of variables of the optimization problem, assuming that the radii of the circles are treated as variables.

Moreover, the proposed approach can be used in solving problems of packing unequal spheres into an arbitrary container.

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