VECTOR CONVEX OPTIMIZATION PROBLEMS: LEXICOGRAPHIC OPTIMALITY AND SOLVABILITY, METHOD OF SOLVING

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Problems of vector optimization

Problems of vector optimization is one of the sections of the modern theory of extreme problems, which has been intensively developing recently. A significant contribution to the development of the theory and methods of vector optimization, including discrete, in Ukraine was made by I.V. Sergienko, V.S. Mihalevisch, V.L. Volkovisch, V.O. Perepelitsa, Yu. Yu. Chervak, A.M. Voronin, L.M. Kozeratskaya, T.T. Lebedeva, N.V. Semenova., T.I. Sergienko, N.K. Makchichko, L.M. Kolechkina, abroad – V.A. Yemelichev, D.P. Pidkopaev, Yu.V. Nikulin, V.V. Podinovsky, V.D. Nogin, V.M. Gavrilov, L.I. Polishchuk, M. Zeleny, K. Mettinen, F. Ruiz, A.P. Verzhbitsky, M. Ergott.

At Uzhhorod University Bryla A.Yu., Chupov S.V. deal with these problems.

Formulation of the problem

$$Z_L(F, X): \quad \max^L \{F(x) | x \in X\}, \tag{1}$$

where $F(x) = (f_1(x), \ldots, f_\ell(x))$, $\ell \geq 2$, $f_k(x) = \langle c_k, x \rangle$, $c_k \in R^n$, $k \in N_\ell = \{1, 2, \ldots, \ell\}$,

$X = \{x \in R^n \mid g^i(x) \leq 0, x \geq 0, i \in N_m\}$, $X \neq \varnothing$, $g^i(x), i \in N_m$, – convex functions.

The set of lexicographic optimal solutions

$$L(F,X)=\{x\in X\mid \upsilon(x,F,X)=\varnothing\}, \tag{2}$$

where $\upsilon(x,F,X)=\{x'\in X\mid \exists j\in N_\ell : f_j(x')>f_j(x)\wedge j=\min\{i\in N_\ell : f_i(x')\neq f_i(x)\}\}$.

or

$$L_i(F,X)=\operatorname{Arg\,max}\{f_i(x): x\in L_{i-1}(F,X), i\in N_\ell, \tag{3}$$

where $\operatorname{Arg\,max}\{\cdot\}$ – is a set of all optimal solutions to the corresponding maximization problem, $L_0(F,X)=X$, $L_\ell(F,X)=L(F,X)$.

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Definition 1. A vector $z \in R^{\ell}$ is called lexicographically positive if its first nonzero component in ascending order of the component indices is positive.

Definition 2. A solution $x^* \in X$ to a problem $Z_L(F, X)$ will be called lexicographically optimal if it is not worse than any other admissible solution $y \in X$ in understanding the relation $\geq^L$, that is, if $F(x^*) - F(y) \geq^L 0$.

Existence of lexicographically optimal solutions

We will analyze the problem $Z_L(F, X)$ taking into account the properties of the recessive cone

$$0^+ X = \left\{ y \in R^n \mid \forall x \in X : x + ty \in X, t \geq 0 \right\} \tag{4}$$

and the cone

$$K^L = \left\{ x \in R^n \middle| Cx >^L 0 \right\} \tag{5}$$

lexicographically ordering the admissible set with respect to the optimization criteria, which we will also call the cone of promising lexicographic directions of the problem $Z_L(F, X)$.

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$$K^L = K_1 \cup K_2 \cup ... \cup K_\ell,$$

where $K_1 = \left\{ x \in R^n \mid c_1 x > 0 \right\},$

$$K_2 = \left\{ x \in R^n \mid c_1 x = 0, c_2 x > 0 \right\}, \qquad (6)$$

$\ldots,$

$$K_\ell = \left\{ x \in R^n \mid c_1 x = 0,\ c_2 x = 0, ..., c_{\ell-1} x = 0,\ c_\ell x > 0 \right\}.$$

Theorem 1. A necessary condition for the existence of lexicographically optimal solutions to the problem $Z_L(F, X)$ is the empty intersection of the cone K^L of promising lexicographic directions and the recessive cone $0^+ X$, that is,

$$K^L \bigcap 0^+ X = \varnothing . \qquad (7)$$

Theorem 2. Let V be a non-empty set of extreme points of a convex closed set X. If V is a bounded set, then the set X has a lexicographic maximum if and only if it is bounded in all lexicographically positive directions.

In our notation, under the conditions of Theorem 2, the set $L(F, X)$ is not empty if and only if condition (7) is satisfied.

Theorem 3. Let the feasible set X of the problem $Z_L(F, X)$ be a closed convex polyhedral set. A necessary and sufficient condition for the existence of lexicographically optimal solutions to this problem is the fulfillment of equality (7)

Optimality conditions for solutions

We will denote a subset $\mathrm{Fr}B$ of boundary points of some set. Let $y \in \mathrm{Fr}\, X$.

$N(y) = \{i \in N_m \mid g_i(y) = 0\}$, $X(y) = \left\{x \in R^n \mid g_i(x) \le 0,\ i \in N(y)\right\}$,

$Q(y) = \{x \in R^n \mid \langle \nabla g_i(y),\ x - y\rangle \le 0,\ i \in N(y)\}$, where $\nabla g_i(y)$ – function gradient $g_i(x)$ at the point $y, i \in N(y)$.

Theorem 4. Let $y \in \mathrm{Fr}\, X$. If $g_i(x), i \in N(y)$, – are continuously differentiable functions, then the relation

$$K^L \bigcap \left(Q(y) - y\right) = \varnothing \qquad (8)$$

is a sufficient condition for the inclusion $y \in L(C, X)$. Moreover, if $\left\{\nabla g_i(y) \mid\ i \in N(y)\right\}$ – is a system of linearly independent vectors, then the relation

$$K_1 \bigcap \left(Q(y) - y\right) = \varnothing \qquad (9)$$

is a necessary condition for the inclusion $y \in L(C, X)$.

Cutting plane method for solving lexicographic vector convex optimization problems

$$Z_L\left(F, X_p\right): \max^L \left\{F(x) \middle| x \in X_p\right\} \qquad (10)$$

$$X_p = \left\{x \in R^n \middle| \left\langle \nabla g^i\left(x^j\right), x - x^j \right\rangle + g^i\left(x^j\right) \le 0,\ x \ge 0, i \in N_m,\ j = 0,1,\ldots, p\right\}, \qquad (11)$$

$$x^j \in R_+^n,\ R_+^n = \left\{x \in R^n \middle| x_i \ge 0, i \in N_n\right\}.$$

Statement 1. The including $X \subset X_p$ is just.

Theorem 5. If a vector function F reaches a lexicographic maximum on a set X_p, then among the points of this maximum there is an extreme point of the set X_p.

Algorithm for solving the problem $Z_L(F, X)$

Initial step. Let $s = 1, k = 0.$ First we select an arbitrary point $x^k \in \mathrm{Fr}\, G$.

Ten we build a polyhedron $X_k = \left\{ x \in R^n \middle| \left\langle \nabla g^i\left(x^k\right), x - x^k \right\rangle + g^i\left(x^k\right) \le 0,\ x \ge 0,\ i \in N_m \right\}$.

1. We will solve the problem

$$\max\left\{ f_s(x) \middle| x \in X_k \right\}.$$

by the dual simplex algorithm. Let $x^{k+1} = \arg\max\left\{ f_s(x) \middle| x \in X_k \right\}$. If $x^{k+1} \in X$, and x^{k+1} – is the only optimal solution on the feasible set X, then $x^{k+1} = \arg\max^L \left\{ F(x) \middle| x \in X \right\}$, insofar as $X \subseteq X_k$. The problem $Z_L(F, X)$ is solved.

Algorithm for solving the problem $Z_L(F,X)$

2. If $x^{k+1} \in X$ and x^{k+1} – is not the only optimal solution on the feasible set X, we believe $\overline{f}_s = f_s\left(x^{k+1}\right)$, $s = s+1$, $X_{k+1} = \left\{x \in X_k \mid f_i(x) = \overline{f}_i, i = 1,2,...,s-1\right\}$ and pass over to step 1. If $x^{k+1} \notin X$ we go to step 3.

3. We define the set $I_{k+1} = \left\{i \left| g^i\left(x^{k+1}\right) > 0\right.\right\}$ of constraint indices of problem $Z_L(F,X)$ which are violated at the point x^{k+1}. We will build a polyhedron X_{k+1}, adding to the constraints describing the set X_k, the inequality

$$\left\langle \nabla g^i\left(x^{k+1}\right), x - x^{k+1}\right\rangle + g^i\left(x^{k+1}\right) \le 0,\ i \in N_{k+1} = \left\{ j \in I_{k+1} \left| g^j\left(x^{k+1}\right) = \max_{i \in I_{k+1}} g^i\left(x^{k+1}\right)\right.\right\}.$$

We get a new multifaceted set

$$X_{k+1} = \left\{x \in X_k \mid \left\langle \nabla g^i\left(x^{k+1}\right), x - x^{k+1}\right\rangle + g^i\left(x^{k+1}\right) \le 0, i \in N_{k+1}\right\}.$$

We go to step 1, believing $k = k+1$.

The convergence of the algorithm

Theorem 6. If the functions $g^i(x),\ i \in N_m,$ are convex, continuously differentiable and the problem $Z_L(F, X)$ has a finite optimal solution, then the sequence of points generated by this algorithm converges to the lexicographically optimal solution to the problem $Z_L(F, X)$.

Conclusions

The issues of existence and lexicographic optimality of solutions of vector convex optimization problems with linear functions of criteria and an unbounded admissible set have been investigated. Based on the analysis of these problems, taking into account the properties of the cones of perspective lexicographic directions, recessive directions and local tents at the boundary points of the feasible set, necessary and sufficient conditions for the existence and lexicographic optimality of solutions of the problems under study have been established. The obtained conditions can be successfully used in the development of algorithms for finding optimal solutions to these lexicographic optimization problems. Based on the ideas of linearization methods and Kelly's cutting planes, a method for finding lexicographically optimal solutions of convex lexicographic problems has been constructed and substantiated.

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Thank you for attention!