On Fuzzy Similarity Relations for Heterogeneous Fuzzy Sets

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Way 1 to define a fuzzy similarity measure

- Our study considers two ways of building a fuzzy similarity measure on homogeneous and heterogeneous fuzzy sets under uncertainty in measurements of membership functions values caused by not statistical in nature and subjective expert assessments.

- Consider the case where a fuzzy scale defines a fuzzy similarity relation or measure. The scale will further referred to as a fuzzy similarity scale. The problem of building a scale is important, for example, in fuzzy clustering under uncertainty due to a verbal description of properties of empirical objects.

Definition 1 A tuple $\mathrm{M} = \langle A; R_1, \ldots, R_n \rangle$ is called a system with relations, where *A* is a non-empty set called a domain (support) of a system with relations, and $R_1, \ldots, R_n$ are the relations given on A.

Let R_i is a k_i-ary relation on A, $i = 1, \ldots, n.$

Definition 2 An ordered sequence of positive integers $\langle k_i \rangle_{i=1}^{n}$ is called type of a system with relations M.

If $A \subseteq R^1$, where R^1 is the set of real numbers, then the system M is called a numerical system with relations (NSR). If A is a set of non-numeric mathematical objects (for example, symbols, vectors, functions), then M is called a mathematical system with relations (MSR).

Let two similar systems are given:

an ESR $\mathrm{M}_1 = \langle E; S_1, \ldots, S_n \rangle$, where $E = \{e_j\}_j$ is a set of ESR objects (elements), $S_1, \ldots, S_n$ are relations given on *E*;

and $\mathrm{M}_2 = \langle A; R_1, \ldots, R_n \rangle$ is a NSR (or MSR), where $A = \{a_j\}_j$ is a set of NSR elements (or MSR), $R_1, \ldots, R_n$ are relations given on A.

A Scale of Measurements

Definition 3 A mapping $f: E \to A$ is called a scale of measurements (in short, a scale) if f is homomorphism, that is,

$$S_i\left(e_1, \ldots, e_j, \ldots, e_{k_i}\right) = R_i\left(a_1, \ldots, a_j, \ldots, a_{k_i}\right),$$

where $\left(e_1, \ldots, e_j, \ldots, e_{k_i}\right) \in E$, $\left(a_1, \ldots, a_j, \ldots, a_{k_i}\right) \in A$, $a_j = f\left(e_j\right)$ are the scale values, and $i \in \{1, \ldots, n\}$.

If the mapping f is bijection, then f is isomorphism.

Let us introduce the definition of a "fuzzy scale" of measurements. Let U be a non-empty set of empirical objects, S_i $(i = 1, \ldots, n)$ is a set of relations on U, L is a subset of real numbers, T is a set of fuzzy subsets on L, R_i $(i = 1, \ldots, n)$ are k_i-ary relations.

Definition 4 A mapping $\mu: U \to T$ is called a fuzzy scale, if $\forall i \in \{1, \ldots, n\}$ and $\forall \left(u_1, \ldots, u_{k_i}\right) \in U^{k_i}$,

$$S_i\left(u_1, \ldots, u_{k_i}\right) = R_i\left(\mu\left(u_1\right), \ldots, \mu\left(u_{k_i}\right)\right),$$

where $\mu\left(u_1\right), \ldots, \mu\left(u_{k_i}\right)$ are the corresponding scale values from T.

Using fuzzy operators for building a fuzzy similarity scale

- The scale is defined as a homomorphic mapping of an ESR onto MSR if relations in the MSR are defined on a set of fuzzy subsets. Defining the scale in representative measurement theory starts with building an ESR model.

Let X be a finite set of objects (elements) of the empirical system and W be a finite set of fuzzy properties of elements defined verbally. The result of measurements of values of membership functions for a set of properties $W=\{w_1, \ldots, w_n\}$, where n is the number of properties, can be a homogeneous or heterogeneous fuzzy set. Membership function is defined as follows: $\mu_W : X \to L_1 \times \ldots \times L_n$, and L_i is some lattice, i.e., $\mu_W(x)=(\mu_1(x),\ldots,\mu_n(x))$, $\mu_i : X \to L_i$, and $i=\{1,\ldots,n\}$. Based on heterogeneous fuzzy sets, it is possible to build models under different types of properties, which are measured by both quantitative and qualitative scales.

Let the property (for convenience, an index i will be omitted below) takes a finite set $T_w = \{t_1, \dots, t_{m(w)}\}$ of verbal values (features, gradations, linguistic terms), where $m(w)$ is the number of these values. If values $\{t_1, \dots, t_{m(w)}\}$ of property $w \in W$ are measured on a certain scale, the result of the property measurement is a homogeneous fuzzy subset of T_w.

If, in verbal measurements, different scale types or scales of same type are applied, while the valid transformations are independent, the measurements results $\{t_1, \dots, t_{m(w)}\}$ of membership function for any $w \in W$ is a heterogeneous fuzzy set.

Definition of fuzzy scale

Let for any $w \in W$ be defined $\tilde{R}: X \times T_w \to [0,1]$ – a fuzzy binary relation on $X \times T_w$, then $\mu_{\tilde{R}}(x,t)$ be a membership function defining this fuzzy relation and $\mu_{\tilde{R}}(x,t) \in [0,1]$, $x \in X$, and $t \in T_w$.

Denote by F_{T_w} a set of all fuzzy subsets of T_w and let $\eta: X \to F_{T_w}$ be a mapping such that $\forall t_j \in T_w \ \forall x \in X \ \ \eta_x(t_j) = \mu_{\tilde{R}}(x,t_j)$. Here and beyond, in similar cases, $j = 1,\dots,m(w)$.

Let for any $x \in X$ $\tilde{D}_x = \left\{ \left(t_j, \eta_x\left(t_j\right) \right) \mid t_j \in T_w \right\}$ be a fuzzy subset from T_w, called a measurement description of $w \in W$. The meaning of $\eta_x(t_j)$ will be considered as degree of truth of statement "x has the meaning t_j of a fuzzy property w". Let $\forall x \in X$ $\tilde{D}_x$ be a homogeneous fuzzy set.

Definition of fuzzy scale

Let $\Omega_{\tilde{D}} = \{\tilde{D}_x \mid x \in X\}$ be a set of fuzzy measurement descriptions. We consider $\Omega_{\tilde{D}}$ as a vector-valued fuzzy set or as a fuzzy type-2 set:

$$\Omega_{\tilde{D}} = \{(x, \vec{\eta}(x)) | x \in X\} = \left\{\left(x, \left(\eta_x(t_1), ..., \eta_x\left(t_{m(w)}\right)\right)\right) | x \in X\right\},$$

where $\eta_x(t_j) \in [0,1], \ j = 1, ..., m(w),$ and $m(w)$ is the number of property values.

For any $x \in X$, measurements of a certain property $w \in W$ are not a real numbers, but are a fuzzy subset of set of values from the corresponding T_w, to which the studying of problems of representation, uniqueness and adequacy is carried out taking into account the structure of fuzzy measurement scale. However, in this case, we change known definition of fuzzy scale: we define fuzzy subsets on a set of verbal values $T_w = \left(t_1, ..., t_{m(w)}\right)$ of a certain property $w,$ and not on a subset of real numbers.

Definition of fuzzy scale

A fuzzy similarity measure on $X \times X$ is a function $\tilde{\tau}(x,y)$, satisfying following conditions: 1) $\forall (x,y) \in X \times X \quad \tilde{\tau}(x,y) \in [0,1]$; 2) $\forall x \in X \quad \tilde{\tau}(x,x) = 1$ (reflexivity); 3) $\forall (x,y) \in X \times X$ $\tilde{\tau}(x,y) = \tilde{\tau}(y,x)$ (symmetry).

A function $\tilde{\tau}(x,y)$ defines a fuzzy set in sense of L. Zadeh. A fuzzy similarity measure $\tilde{\tau}(x,y)$ should be determined taking into account the results of measuring for all values of fuzzy property $w \in W$ for any $x \in X$, that is, by aggregating fuzzy logical operators.

A method of calculating values $\tilde{\tau}(x,y)$ is determined both by type of scale of measuring values of $w \in W$ for objects of an empirical system, and by theoretical-set interpretation of semantic link AND and OR in form of fuzzy logical operators.

A fuzzy analogue of semantic link AND is triangular norm and fuzzy analogue of OR is an operator of triangular conorm. In fuzzy sets theory, there are many variants for choosing of norm and conorm operators. We use fuzzy operators of Zadeh norms and conorms:

$$T(a,b) = \min(a,b)\,,\ S(a,b) = \max(a,b),\ a,b \in [0,1],$$

and fuzzy operators of norms and conorms (Ia. Lukasiewicz)

$$T(a,b) = \max(0, a+b-1)\,,\ S(a,b) = \min(1, a+b),\ a,b \in [0,1].$$

Definition of fuzzy similarity measure

Let $\tilde{\tau}(\cdot,\cdot)$ be a fuzzy similarity measure defined on $X \times X$ and $\tilde{\tau}^*(\cdot,\cdot)$ be a fuzzy similarity measur defined on $F_{T_w} \times F_{T_w}$. The Homomorphic mapping $\eta : X \to F_{T_w}$ determines a fuzzy measurement sc $w \in W$. However, the existence of a scale, i.e., a homomorphism of η mapping we must prove. Denote by F_X a set of all subsets of *X*. Let $\delta : T_w \to F_X$ is the mapping such that $\forall x \in X \ \forall t_j \in T_w$ $\delta_{t_j}(x) = \mu_{\tilde{R}}(x, t_j)$ and let $\tilde{E}_j = \left\{ \left(x, \delta_{t_j}(x) \right) \mid x \in X \right\}$ be a fuzzy subset of *X*, called a meaning of $t_j \in T_w$.

Obviously, a condition for similarity $(\tilde{\tau}(x,y) > 0)$ of $x, y \in X$ is an existence of at least one commo property, i.e.:

$$\tilde{\tau}(x,y) > 0 \Leftrightarrow \left(\exists t_j \in T_w \right) : \left(x \in Supp\, \tilde{E}_j \right) \wedge \left(y \in Supp\, \tilde{E}_j \right). \qquad (1)$$

Since $\delta_{t_j}(x) = \eta_x(t_j)$, then it is from (1) follows that intersection of supports, i.e., $Supp\ \tilde{D}_x \cap Supp\ \tilde{D}_y$ also a condition of similarity of empirical objects. If the case $Supp\ \tilde{D}_x \cap Supp\ \tilde{D}_y = \varnothing$ is valid, then $\tilde{\tau}(x,y) = 0$. Thus,

$$\tilde{\tau}(x,y) > 0 \Leftrightarrow \left(\exists t_j \in T_w \right) : \left(\eta_x(t_j) > 0 \right) \wedge \left(\eta_y(t_j) > 0 \right). \qquad (2)$$

Definition of fuzzy similarity measure

Considering (2), we define $\varphi(\tilde{D}_x, \tilde{D}_y)$ as follows:

$$\varphi\left(\tilde{D}_x, \tilde{D}_y\right) = \left(\left(\eta_x\left(t_1\right) > 0\right) \wedge \left(\eta_y\left(t_1\right) > 0\right)\right) \vee \ldots \vee \left(\left(\eta_x\left(t_m\right) > 0\right) \wedge \left(\eta_y\left(t_m\right) > 0\right)\right). \quad (3)$$

Replacing in (3) inequalities with corresponding values of membership function, and logical operators with triangular norms T and conorms *S*, we get:

$$\tilde{\varphi}\left(\tilde{D}_x, \tilde{D}_y\right) = \underset{t_j \in T_w}{S}\left(T\left(\eta_x\left(t_j\right), \eta_y\left(t_j\right)\right)\right),\ \tilde{D}_x, \tilde{D}_y \in \Omega_{\tilde{D}}. \quad (4)$$

Obviously, $\tilde{\varphi}(\tilde{D}_x, \tilde{D}_y) \in [0,1]$. Symmetry $\tilde{\varphi}(\tilde{D}_x, \tilde{D}_y)$ follows from the symmetry of operators of norm and conorm operators.

If (4) satisfies reflexivity condition, i.e., if the condition, $\tilde{\varphi}(\tilde{D}_x, \tilde{D}_y) = 1$ is met, then we define a fuzzy similarity measure on $F_{T_w} \times F_{T_w}$ as

$$\tilde{\tau}^*\left(\tilde{D}_x, \tilde{D}_y\right) = \tilde{\varphi}\left(\tilde{D}_x, \tilde{D}_y\right).$$

Therefore,

$$\tilde{\tau}^*\left(\tilde{D}_x, \tilde{D}_y\right) = \underset{t_j \in T_w}{S}\left(T\left(\eta_x\left(t_j\right), \eta_y\left(t_j\right)\right)\right),\ \tilde{D}_x, \tilde{D}_y \in \Omega_{\tilde{D}}. \quad (5)$$

Such a method of defining a fuzzy similarity measure, which is an analogue of method of calculating a similarity measure for crisp properties based on direct replacement of logical operators with their fuzzy analogues, we call as "direct" method.

A Fuzzy Similarity Scale

We suppose, that $\tilde{\tau}(x,y)=\tilde{\tau}^*\left(\tilde{D}_x,\tilde{D}_y\right)$. If reflexivity condition $\tilde{\tau}^*\left(\tilde{D}_x,\tilde{D}_x\right)$=1 is met, then formula (5) defines a fuzzy similarity measure.

Obviously, mapping $\eta: X \to \Omega_{\tilde{D}}$ gives rise to an equivalence relation θ, i.e.:

$$x\theta y \Leftrightarrow \tilde{D}_x = \tilde{D}_y,$$

that under the reflexivity of $\tilde{\tau}^*\left(\tilde{D}_x,\tilde{D}_y\right)$ is a congruence relation:

$$\left(x\theta x'\right)\wedge\left(y\theta y'\right)\Rightarrow\tilde{\tau}\left(x,y\right)=\tilde{\tau}\left(x',y'\right).$$

Thus, we define two systems with relations: an ESR, $\mathrm{M}_1=\langle X,\theta,\tilde{\tau}\rangle$ and a MSR, $\mathrm{M}_2=\left\langle\Omega_{\tilde{D}},=,\tilde{\tau}^*\right\rangle$. A tuple $\langle\mathrm{M}_1,\mathrm{M}_2,\eta\rangle$ we call a fuzzy scale for measuring a partial fuzzy similarity measure (further – a fuzzy similarity scale).

Representation theorem

- We formulate a representation theorem determining conditions for existence of measurement scale of a fuzzy similarity measure (5).
- Proving representation theorem, it is believed that an absolute measurement scale is used, while only identical transformations are permissible.
- **Theorem 1** A reflexivity condition for fuzzy similarity measure (5) is in generally not satisfied.

Representation theorem

If we use in (5) Zadeh norm and conorm operators, we get

$$\forall x \in X \;\exists t_j : \eta_x(t_j) = 1. \qquad (6)$$

Therefore, measurement results must consist of normal fuzzy sets. In general, this condition may not be met.

The Lukasevich norm operator cannot be used in (5), since in case where the sum of values of membership functions is less than 1, value of this norm is zero, then

$$\tilde{\tau}^*(\tilde{D}_x, \tilde{D}_x) = 0,$$

and the reflexivity condition is not fulfilled.

If in (5) we use Zadeh norm and Lukasevich conorm, then we get

$$\tilde{\tau}^*(\tilde{D}_x, \tilde{D}_x) = \min\left\{1, \sum_{t_j \in T_w} \eta_x(t_j)\right\}.$$

In this case, the reflexivity condition is as follows:

$$\sum_{t_j \in T_w} \eta_x(t_j) \geq 1 \;\forall x \in X, \qquad (7)$$

which in practice would not always be fulfilled.

An Adequacy Problem

Let us study an adequacy of fuzzy similarity measure.

Theorem 2 Fuzzy similarity measure (5) is not adequate while measuring values of membership function of a fuzzy property $w \in W$ in ratio, interval, and order scales.

Proof. Truthiness of theorem falls out of absence of invariance of conditions (6) and (7) for measurements in ratio and interval scales. Condition (7) cannot be used in order scale because sum operation is invalid. It is obvious that condition (6) violates in monotonic transformations permissible in order scale.

The theorem is proved.

Thus, fuzzy similarity measure in "direct" method of determining (5), is generally unsuitable for constructing an adequate formal model, while component of it is a fuzzy similarity relation.

Way 2 to define a fuzzy similarity measure

Let X be a finite set of empirical system objects (elements), $W=\{w_1,\ldots,w_n\}$ is a finite set of fuzzy properties of X defined verbally. Let property $w_i \in W$ takes a finite set of verbal values $T_{w_i}=\{t_1^i,\ldots,t_{m(w_i)}^i\}$, where $m(w_i)$ is the number of values of $w_i \in W$.

Definition 5 A linguistic correlation coefficient (LCC, K_{lingv}) and a partial linguistic correlation coefficient (k_i) we call, respectively,

$$K_{\text{lingv}}(x,y)=\frac{1}{n}\sum_{i=1}^{n}k_i(x,y), \tag{8}$$

$$k_i(x,y)=\tilde{\tau}_i^*\left(\tilde{D}_x^i,\tilde{D}_y^i\right)=\left(\sum_{j=1}^{m(w_i)}\min\left(\eta_x(t_j^i),\eta_y(t_j^i)\right)\Big/\sum_{j=1}^{m(w_i)}\max\left(\eta_x(t_j^i),\eta_y(t_j^i)\right)\right), \tag{9}$$

where $\tilde{\tau}_i^*\left(\tilde{D}_x^i,\tilde{D}_y^i\right)$ is a partial similarity measure in MSR of $w_i \in W$, and $\tilde{D}_x^i$, $\tilde{D}_y^i$ are measurements of $w_i \in W$ for elements $x,y \in X$, respectively, n is the total number of properties; $\eta_x(t_j^i)$ determines a measure of belonging of value t_j^i to w_i of $x \in X$, $j \in \{1,\ldots,m(w_i)\}$, $i \in \{1,\ldots,n\}$.

Way 2 to define a fuzzy similarity measure

Therefore, a partial LCC determines a value of a partial similarity measure on a set of empirical objects. Obviously, when measuring values of membership functions in an absolute scale

$$k_i(x,x)=1,\ \ k_i(x,y)=k_i(y,x);$$

$$K_{\text{lingv}}(x,x)=1,\ \ K_{\text{lingv}}(x,y)=K_{\text{lingv}}(y,x);$$

and

$$k_i(x,y)\in[0,1],\ K_{\text{lingv}}(x,y)\in[0,1];$$

that is, according to formulas (8) and (9), a fuzzy similarity measure on $X \times X$: $\tilde{\tau}(x,y)=K_{\text{lingv}}(x,y)$ is determined. Unlike "direct" method of determining of similarity measure (5) in the case of measurements of values of membership functions on an absolute scale, in order to ensure reflexivity condition of fuzzy similarity measure with help of LCC, no restrictions on value of membership functions of fuzzy objects properties are necessary. Since min and max operators are valid for order, interval, ratio, and absolute scale, such theorems are valid.

Way 2 to define a fuzzy similarity measure

Theorem 3 Partial fuzzy similarity measure of (9) is invariant when measuring values of membership function of a fuzzy qualitative property $w_i \in W$ in ratio scales.

Theorem 4 When measuring values of membership functions of a fuzzy qualitative property in an order scale and an interval one, there exist a permissible monotonic transformation $\psi\left(\eta_x(t_j^i)\right)$, $t_j^i \in T_{w_i}$, leading to invariance of values of membership function of a partial fuzzy similarity measure (9), $i \in \{1, \ldots, n\}$.

Invariance of partial similarity measures leads to invariance of LCC, using which as a similarity measure provides an adequacy of formal model of ESR as example in fuzzy clustering.

Example of LCC Calculation

Let the manager of the enterprise personnel department be faced with the problem of selecting several of the best representatives from the candidates for vacant positions of engineers of the department producing equipment, for example, for a new model of the aircraft.

To determine the best applicants, it is necessary to evaluate their competitiveness according to the totality of two fuzzy qualitative characteristics (w_1, w_2), where w_1 – the work experience, w_2 – the level of qualification. Characteristics w_1, w_2 are fuzzy multidimensional expert assessments measured in order scales.

We believe that the description of the work experience has two gradations $w_1 = \left(t_1^1,\, t_2^1\right)$, where t_1^1 be the work experience according to the profile of the enterprise; t_2^1 be the work experience on a computer with programmable logic matrices. We assume that the qualification level has three gradations $w_2 = \left(t_1^2,\, t_2^2,\, t_3^2\right)$, where t_1^2 – knowledge of foreign languages, t_2^2 – knowledge of a specify set of programming languages, t_3^2 – the duration of work in leading companies in the industry. Let $m = 2$ be the number of vacancies, $X = \{x_i\}_{i=1}^{p}$ is the set of applicants, $p = 4$ be the number of applicants.

Example of LCC Calculation

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Let the manager give the following gradations t_1^1, t_2^1 of characteristic w_1 for applicants: $x_1 - (0,6;0,9)$, $x_2 - (0,5;0,4)$, $x_3 - (0,8;0,7)$, $x_4 - (0,4;0,5)$. The ideal applicant has the maximum value of gradations of characteristics, $x_{id} - (1;1)$. The variation series of manager assessments of gradation t_1^1, t_2^1 of the characteristic w_1 are respectively:

$$\left(t_1^1(x_1), t_1^1(x_2), t_1^1(x_3), t_1^1(x_4), t_1^1(x_{id})\right) = (0,6;0,5;0,3;0,4;1),$$

$$\left(t_2^1(x_1), t_2^1(x_2), t_2^1(x_3), t_2^1(x_4), t_2^1(x_{id})\right) = (0,9;0,4;0,7;0,5;1).$$

Example of LCC Calculation

Obviously, the corresponding rank vector of the assessments of t_1^1 (ordered in ascending order) for the sequence $(x_1, x_2, x_3, x_4, x_{id})$ has the form: $\left(r_1^1(x_1), r_1^1(x_2), r_1^1(x_3), r_1^1(x_4), r_1^1(x_{id})\right) = (4;3;1;2;5)$. Similarly, the rank vector of the assessments of $t_2^1 - \left(r_2^1(x_1), r_2^1(x_2), r_2^1(x_3), r_2^1(x_4), r_2^1(x_{id})\right) =$ $(4;1;3;2;5)$.

Since rank values are invariant to permissible (monotonic) transformations in the order scale, the value of the membership functions of fuzzy characteristics w_1 , w_2 is calculated by dividing the rank assessments by a maximum rank value of 5. Thus, for the ordered totality $(x_1, x_2, x_3, x_4, x_{id})$, we obtain, respectively, the values of the membership functions to the assessments t_1^1 and t_2^1:

$$\left(\eta_{x_1}(t_1^1), \eta_{x_2}(t_1^1), \eta_{x_3}(t_1^1), \eta_{x_4}(t_1^1), \eta_{x_{id}}(t_1^1)\right) = (0,8;0,6;0,2;0,4;1),$$

$$\left(\eta_{x_1}(t_2^1), \eta_{x_2}(t_2^1), \eta_{x_3}(t_2^1), \eta_{x_4}(t_2^1), \eta_{x_{id}}(t_2^1)\right) = (0,8;0,2;0,6;0,4;1).$$

Example of LCC Calculation

Let the manager give the following gradations t_1^2, t_2^2, t_3^2 of characteristic w_2 for applicants: $x_1 - (0,8;0,7;0,3)$, $x_2 - (0,3;0,6;0,8)$, $x_3 - (0,5;0,9;0,2)$, $x_4 - (0,6;0,8;0,6)$. The ideal applicant has the maximum value of gradations of characteristics w_2: $x_{id} - (1;1;1)$.

The variation series of gradation manager assessments of t_1^2, t_2^2, t_3^2 for w_2 have the form:

$$\left(t_1^2(x_1), t_1^2(x_2), t_1^2(x_3), t_1^2(x_4), t_1^2(x_{id})\right) = (0,8;0,3;0,5;0,6;1),$$

$$\left(t_2^2(x_1), t_2^2(x_2), t_2^2(x_3), t_2^2(x_4), t_2^2(x_{id})\right) = (0,7;0,6;0,9;0,8;1),$$

$$\left(t_3^2(x_1), t_3^2(x_2), t_3^2(x_3), t_3^2(x_4), t_3^2(x_{id})\right) = (0,3;0,8;0,2;0,6;1).$$

It is not difficult to show that the corresponding rank vector of assessments of t_1^2 for the sequence $(x_1, x_2, x_3, x_4, x_{id})$ is: $\left(r_1^2(x_1), r_1^2(x_2), r_1^2(x_3), r_1^2(x_4), r_1^2(x_{id})\right) = (4;1;2;3;5)$.

Similarly, the rank assessment vector for t_2^2 –

$\left(r_2^2(x_1), r_2^2(x_2), r_2^2(x_3), r_2^2(x_4), r_2^2(x_{id})\right) = (2;1;4;3;5)$, and

$\left(r_3^2(x_1), r_3^2(x_2), r_3^2(x_3), r_3^2(x_4), r_3^2(x_{id})\right) = (2;4;1;3;5)$ is the ranking vector for t_3^2.

Example of LCC Calculation

The value of the membership functions of fuzzy gradations of t_1^2, t_2^2, t_3^2 is calculated by dividing the rank assessments by the maximum rank value of 5. Thus, for an ordered sequence $\left(x_1, x_2, x_3, x_4, x_{id}\right)$, we get the values of the membership functions of the assessments for t_1^2, t_2^2, t_3^2:

$$\left(\eta_{x_1}(t_1^2), \eta_{x_2}(t_1^2), \eta_{x_3}(t_1^2), \eta_{x_4}(t_1^2), \eta_{x_{id}}(t_1^2)\right) = \left(0,8; 0,2; 0,4; 0,6; 1\right),$$

$$\left(\eta_{x_1}(t_2^2), \eta_{x_2}(t_2^2), \eta_{x_3}(t_2^2), \eta_{x_4}(t_2^2), \eta_{x_{id}}(t_2^2)\right) = \left(0,4; 0,2; 0,8; 0,6; 1\right),$$

$$\left(\eta_{x_1}(t_3^2), \eta_{x_2}(t_3^2), \eta_{x_3}(t_3^2), \eta_{x_4}(t_3^2), \eta_{x_{id}}(t_3^2)\right) = \left(0,4; 0,8; 0,2; 0,6; 1\right).$$

To evaluate the measure of similarity of the applicant x_p, $p \in \{1, 2, 3, 4\}$, with the ideal element x_{id} we use $K_{\text{lingv}}(x_p, x_{id})$, calculated as follows

$$K_{\text{lingv}}(x_p, x_{id}) = \frac{1}{n} \sum_{i=1}^{n} k_i(x_p, x_{id}),\ k_i\left(x_p, x_{id}\right) = \tilde{\tau}_i^*\left(\tilde{D}_{x_p}^i, \tilde{D}_{x_{id}}^i\right),$$

$$\tilde{\tau}_i^*\left(\tilde{D}_{x_p}^i, \tilde{D}_{x_{id}}^i\right) = \left(\sum_{j=1}^{m(w_i)} \min\left(\eta_{x_p}(t_j^i), \eta_{x_{id}}(t_j^i)\right) \middle/ \sum_{j=1}^{m(w_i)} \max\left(\eta_{x_p}(t_j^i), \eta_{x_{id}}(t_j^i)\right) \right).$$

Example of LCC Calculation

After simple calculations according to the above formulas, we get

$$k_1(x_1, x_{id}) = (0,8 + 0,8)/2 = 0,80; \;\; k_2(x_1, x_{id}) = (0,8 + 0,4 + 0,4)/3 = 0,53;$$

$$K_{\text{lingv}}(x_1, x_{id}) = (0,80 + 0,53)/2 = 0,67;$$

$$k_1(x_2, x_{id}) = (0,6 + 0,2)/2 = 0,40; \;\; k_2(x_2, x_{id}) = (0,2 + 0,2 + 0,8)/3 = 0,40;$$

$$K_{\text{lingv}}(x_2, x_{id}) = (0,40 + 0,40)/2 = 0,40;$$

$$k_1(x_3, x_{id}) = (0,2 + 0,6)/2 = 0,40; \;\; k_2(x_3, x_{id}) = (0,4 + 0,8 + 0,2)/3 = 0,47;$$

$$K_{\text{lingv}}(x_3, x_{id}) = (0,40 + 0,47)/2 = 0,44;$$

$$k_1(x_4, x_{id}) = (0,4 + 0,4)/2 = 0,40; \;\; k_2(x_4, x_{id}) = (0,6 + 0,6 + 0,6)/3 = 0,60;$$

$$K_{\text{lingv}}(x_4, x_{id}) = (0,6 + 0,4)/2 = 0,50.$$

According to calculations,

$$K_{\text{lingv}}(x_1, x_{id}) > K_{\text{lingv}}(x_4, x_{id}) > K_{\text{lingv}}(x_3, x_{id}) > K_{\text{lingv}}(x_2, x_{id}).$$

Thus, LCC assessments of the competitiveness of applicants, which use expert assessments of multidimensional fuzzy characteristics measured in order scales, show that one of the two vacancies should be offered to the applicant x_1, and the second vacancy should be offered to the applicant x_4.

Conclusions

- Our study concerns the features of combinatorial optimization problems on fuzzy sets under multidimensional qualitative and quantitative information. Uncertainty caused by measurements of membership functions values of fuzzy sets in different scales leads to considering the base problems studied in representative measurement theory, i.e., a presentation problem, uniqueness one and an adequacy.
- We offer a concept of fuzzy similarity scale.
- An inadequacy of building a fuzzy similarity scale based on fuzzy logic operators we prove considering the representative measurement theory positions.
- A concept of a linguistic correlation coefficient is introduced. Conditions of its adequacy in different scales of measuring of empirical objects properties, i.e., order, ratio, intervals, and absolute scales (according to Stevens' Classification) are derived.
- As further step of research, based on linguistic correlation coefficient, a fuzzy binary relation of difference on homogeneous and heterogeneous fuzzy sets will be determined.

Thank you for your attention!