Magic type labeling of graphs in linear ordering problems

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A Dynamic Survey of Graph Labeling

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We consider only finite undirected graph without loops and multiple edges. We denote $G=(V,E)$ as a graph, where V is the set of vertices, and E is the set of edges.

Distance magic labeling of graph $G=(V,E)$ of order p is a bijection $f: V(G)\to\{1, 2, \ldots, p\}$, for which there exists an integer number k, such for every vertex u equality $k=\sum_{v\in N(u)} f(v)$, is fair, where $N(u)$ is a set of vertices of graph G adjacent with u. Number k is called *magic constant of labeling* f, and a graph, which allows such labeling is *distance magic*.

Problem 1. Finding the conditions for the existence of distance magic labeling of the graph G, which is not isomorphic to $K_{1,2,\ldots,2}$

Edge labeling φ of graph G is magic, if φ is an injection $E(G)$ into N and the weight of the vertex does not depend on the choice of the vertex. Graph, which allows magic labeling, is called *magic*. We define a constant value of index as *magic constant* and denote as μ. Magic power $\mu(\varphi)$ of labeling φ is maximum of the label, used in φ, and a magic power $\mu(G)$ of graph G is minimum of the powers of the magic labelinges, which the graph allows. If the graph G is not magic, then $\mu(G)=0$ by definition.

Problem 2. Identifying the property of graph, which allows finding of necessary and/or sufficient magic conditions.

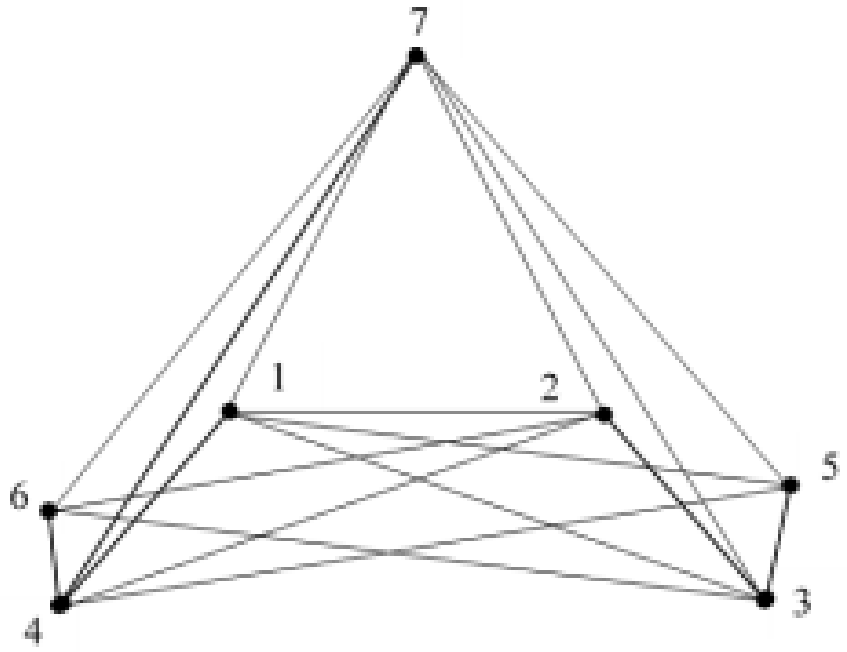
The total vertex-magic labeling of a graph $G=(V,E)$ of order p and size q is denoted as a bijection $g: V(G)\to\{1, 2,..., p+q\}$, for which there exists such a constant k, that for each vertex $u\in V$ the equality holds $g(u)+\sum_{v\in N(u)} g(uv)=k$. The number k is called the magic constant, the sum $g(u)+\sum_{v\in N(u)} g(uv)=w_t(u)$ is called the weight of the vertex u, and the graph G allowing the labeling g is called total vertex-magic.

Problem 3. Determining the types of graph, which allows total vertex-magic labeling.

Theorem 1. If $G \cong (K_{p-1} - M) + K_1$ is a graph of odd order p with $\Delta(G) = p - 1$, where M is perfect matching in K_{p-1}, then $G \cong K_{1,2,\ldots 2}$.

Corollary. Graph $K_{1,2,\ldots,2}$ is the distance magic graph.

Example. Suppose $G \cong (K_6 - M) + K_1$(fig.1). Let us set vertex labeling of G, as it is shown in figure 1. The labeling is distance magic with magic constant $k = 21$. Let us identify the vertices with their labels. Then we divide the set of vertices $V(G)$ into subsets {1,6}, {2,5}, {3,4}, {7}. They are independent, moreover, these subsets are particles of the graph $K_{1,2,2,2}$, such that $G \cong K_{1,2,2,2}$.



Theorem 2. If $G=(V,E)$ is a distance magic graph different from $K_{1,2,2,\ldots,2}$, then the set V of can be partitioned into sets V_1, V_2, …, V_p, in such way that $|V_i|>1$ and $G(V_i)$ is the empty graph for every $i=1,2,\ldots,p$.

Proof. According to the conditions $G = (V, E)$ is a graph other than $K_{1,2,\ldots,2}$, and G allows distance magic labeling f. Let G be a connected graph, in other case every component of the connectivity of this graph should be investigated in a similar way separately.

Any graph can be decomposed into subgraphs, each of which is a click or an empty graph. Therefore, we examine the graph G with the partition of the set of vertices $V(G)$ on clicks and independent sets. Let $V_1, V_2, \ldots, V_p$ be the partitions, and V_1 is a click of order n, and $V_2, V_3, \ldots, V_p$ are independent sets. In addition, we will assume that n is a minimum number for which there is a specified partition, otherwise the partition process can be continued.

Let denote the vertices $G(V_i)$ as $\{u_1^i, u_2^i, \ldots, u_{|V_i|}^i\}$ and the degree of vertex u_j^i in graph G as $deg_G(u_j^i)$, where $i = 1, 2, \ldots, p, j = 1, 2, \ldots, |V_i|$.

If S is a sum of vertex labels in $G(V_1)$ and $deg_G(u_l^1) = deg_G(u_m^1) = n - 1$, where $u_l^1, u_m^1 \in V_1$, then we obtain the following weights for these vertices: $\omega_G(u_l^1) = S - f(u_l^1)$ and $\omega_G(u_m^1) = S - f(u_m^1)$. Here we have $f(u_l^1) = f(u_m^1)$. Thus, the degrees in a graph G less than $n - 1$ of vertex from set $\{u_1^1, u_2^1, \ldots, u_{|V_1|}^i\}$ must exceed $n - 1$, otherwise the condition of being magic is violated. But in this case you can get a new partition into subsets, each of which is empty. Similar considerations can be performed if there is more than one click in the partition.

Suppose, that among $V_1, V_2, \ldots, V_p$, there are at least two sets of degree one, for example, $|V_1| = |V_2| = 1$. Then vertices $u_1^i \in V_i$ $(i = 1, 2)$ must be adjacent to every vertex of the corresponding set $V - V_i$ in graph G. According to theorem 1, the graph G is not distance magic. We came to a contradiction with the condition of the theorem 4.

If there is only one set V_i from $|V_i| = 1$, then its only vertex $u_1^i \in V_i$ is adjacent to every vertex of the set $V - V_i$ in graph G. According to theorems 2 and 3, and the corollary 1, we get $G \cong K_{1,2,2,2}$, which contradicts the condition of the theorem 4.

The theorem has been proved.

Corollary. Let $G=(V,E)$ be a distance magic *r*-regular graph of order *n*. Then for the power of any independent set V_i in G double inequality

$$1<|V_i|\leq \frac{\lambda n}{\lambda - r}$$

is fair, where λ is the minimum eigenvalue of the adjacency matrix G.

Let $G = (V, E)$ be a graph with p vertices and q edges, i.e. (p, q) - the graph with edge labeling φ. Let numbers $x_1, ..., x_q$ form a set of edge labels of G. Then we denote the vertices of G as $u_1, u_2, ..., u_p$. It is obvious, G is the magic graph with magic constant μ if and only if, the system

$$\varphi^*(u_i) = \mu \qquad (i = 1, 2, ..., p)$$

of p linear equations with $q+1$ unknowns $x_1, ..., x_q, \mu$ has a solution on the set of positive integer numbers. If this system has no solution, it means that the graph G is not magic. If there exists such a solution, then it corresponds to the magical labeling of the graph G. In this case, there is an unlimited family of magical labelings if this graph, and among them there is one that will give the value $\mu(G)$. The matrix record the system of linear equation has the form

$$A(G)X = \mu I,$$

where $A(G)$ is the incidence matrix of graph G, $X^T = (x_1, x_2, ..., x_q)$, I –column-matrix with p rows, all elements of which equal one.

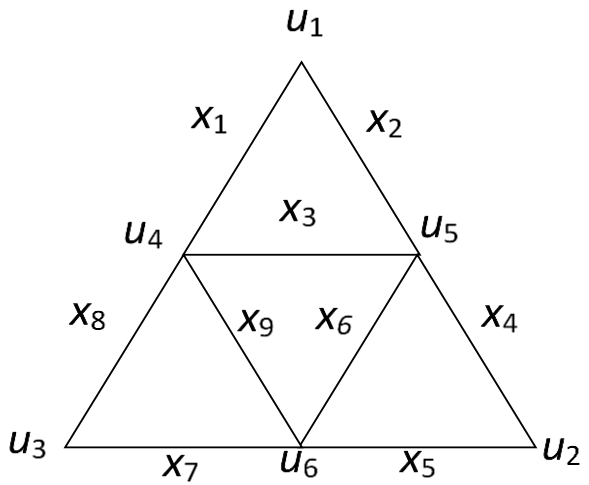
Example. Determine the magical power of the graph *G*, shown in the figure 2.

We set the edge labeling of the graph *G*, as it is shown in figure 1. Suppose, that *G* is a magic with magic constant μ. Then the system of linear equations for the graph looks like

$\varphi^*(u_i)=\mu$, where $i=1, 2, \ldots, 9$ or

$$\begin{cases} x_1 + x_2 = \mu, \\ x_4 + x_5 = \mu, \\ x_7 + x_8 = \mu, \\ x_1 + x_3 + x_8 + x_9 = \mu, \\ x_2 + x_3 + x_4 + x_6 = \mu, \\ x_5 + x_6 + x_7 + x_9 = \mu. \end{cases}$$

$$x_3 + x_6 + x_9 = 0.$$



Lemma 1. If graph G allows magic labeling with magic constant μ, then

$$\mu \geq \Delta(G) \quad \text{and} \quad \mu(G) \geq \left\lceil \frac{\mu}{\delta(G)} \right\rceil.$$

Proof. Weight of vertex μ of maximum degree of graph G is a sum of edge labels, each of which is not less than one 1. Then $\mu \geqslant \Delta(G)$. The equal sign is possible only for the graph K_2. On the other hand, to the vertex of the last degree $\delta(G)$ labels are incidental, which give the sum of μ. Amoung them there is a label which not less that $\mu\ \delta(G)$. Thus, $\mu(G) \geqslant \lceil \mu\ \delta(G) \rceil$.

Lemma has been proved.

Corollary. if G is a magic graph, then $\mu(G) \geq \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$.

Lemma 2. Let G be a graph of order p, which allows magic labeling with magic constant μ. Then the number $\mu\pi$ is even.

Proof. Suppose, that the graph G of order p allows magic labeling with magic constant μ. We denote the sum of all edges of graph G as S. Every vertex from p has a weight μ , then product μ represents twice the amount of labels. Thus, $\mu p = 2S$.

Lemma has been proved.

Corollary 4. If graph G of odd order allows magic labeling with magic constant μ, then $\mu \equiv 0(mod 2)$.

For r-regular graph $G = (V, E)$ of order p and size q with total vertex magic labeling g and magic constant k there is dual labeling g', defined as follows $g'(u) = p + q + 1 - g(u)$ and $g'(uv) = p + q + 1 - g(uv)$ for any $u \in V$, $uv \in E$. We denote magic constant for g' as k' then $k' = k - (r + 1)$.

The graph $G + x = (V \bigcup \{x\}, E^*)$, where $x \notin E(G)$ is obtained from the graph $G = (V, E)$ adding a vertex x and all edges, which are connecting the vertex to all vertices of set $V(G)$.

Let G be a regular graph of order p and size q with total vertex labeling g and $V = \{u_1, u_2, ..., u_p\}$. For the graph $G + x$ we consider such a total labeling g^*, that $g^*(u_i) = g(u_i)$, $g^*(u_iu_j) = g(u_iu_j)$, $g^*(u_ix) = p + q + i$, $g^*(x) = 2p + q + 1$ for any $u_i \in V(G)$, $u_iu_j \in E(G)$, where $i \neq j, i, j \in \{1, 2, ..., p\}$. We obtain the following weights for the vertices of graph $G + x$: $\omega_t(u_i) = k + p + q + i$, $\omega_t(x) = \frac{1}{2}(3p^2 + 5p) + q(p + 1) + 1$. As we can see, the weights of vertices u_i in graph $G + x$ at different values i are different. If we suppose, that there is $i \in \{1, 2, ..., p\}$ at which $\omega_t(u_i) = \omega_t(x)$, then we obtain $k + i = \frac{2}{3}p(p + 1) + pq + 1$. This is impossible since $k + i \neq const$. In this case, the labeling g^* is called total vertex antimagic.

We have proved the following theorem.

Theorem 5. If G is a regular total vertex magic graph, then $G + x$ allows total vertex antimagic labeling.

It is easy to establish the relationship between the magical and total vertex-antimagic labelings of 2-regular graphs.

Theorem 6. Let G be a 2-regular magic graph of order p and size q with a set of edge labeles $\{1, 2, ..., q\}$. Then G allows total vertex antimagic labeling.

Proof. Let φ be a magic labeling of 2-regular graph G of order p and size q, where a set of edge labels $\{1, 2, ..., q\}$, and μ is his magic constant and $V = \{u1, u2, ..., up\}$. We denote total labeling g^* of graph G as the following: $g^*(ui, uj) = \varphi(ui, uj)$, $g^*(ui) = q + i$, for every $ui \in V(G)$, $ui, uj \in E(G)$, where $i \neq j$, $i, j \in \{1, 2, ...p\}$. Let's calculate the weights of the vertices when labeling $g^* : \omega t(ui) = \mu + q + i$. They are all different, therefore, g^* is total vertex-antimagic labeling.

The theorem has been proved.

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Thanks for
your attention.